Fixed point theorems for Boolean networks expressed in terms of forbidden subnetworks

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Abstract

We are interested in fixed points in Boolean networks, *i.e.* functions f from $\{0,1\}^n$ to itself. We define the subnetworks of f as the restrictions of f to the subcubes of $\{0,1\}^n$, and we characterizes a class $\mathcal{F}$ of Boolean networks satisfying the following property: Every subnetwork of f has a unique fixed point if and only if f has no subnetwork in $\mathcal{F}$. This characterization generalizes the fixed point theorem of Shih and Dong, which asserts that if for every x in $\{0,1\}^n$ there is no directed cycle in the directed graph whose the adjacency matrix is the discrete Jacobian matrix of f evaluated at point x, then f has a unique fixed point. Then, denoting by $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) the networks whose the interaction graph is a positive (resp. negative) cycle, we show that the non-expansive networks of $\mathcal{F}$ are exactly the networks of $\mathcal{C}^+\cup\mathcal{C}^-$; and for the class of non-expansive networks we get a "dichotomization" of the previous forbidden subnetwork theorem: Every subnetwork of f has at most (resp. at least) one fixed point if and only if f has no subnetworks in $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) subnetwork. Finally, we prove that if f is a conjunctive network then every subnetwork of f has at most one fixed point if and only if f has no subnetworks in $\mathcal{C}^+$.

Keywords: Boolean network, fixed point, feedback circuit

1. Introduction

A function f from $\{0,1\}^n$ to itself is often seen as a Boolean network with n components. On one hand, the dynamics of the network is described by the iterations of f; for instance, with the synchronous iteration scheme, the dynamics is described by the recurrence $x^{t+1}=f(x^t)$. On the other hand, the "structure" of the network is described by a directed graph $G(f)$: The vertices are the n components, and there exists an arc from j to i when the evolution of the ith component depends on the evolution of the jth one.

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Boolean networks have many applications. In particular, from the seminal works of Kauffman [7] and Thomas [22], they are extensively used to model gene networks. In most cases, fixed points are of special interest. For instance, in the context of gene networks, they correspond to stable patterns of gene expression at the basis of particular biological processes.

Importance of fixed point leads researchers to find conditions for the existence and the uniqueness of a fixed point. Such a condition was first obtained by Robert [16], who proved that *if $G(f)$ has no directed cycle, then f has a unique fixed point*. This result was then generalized by Shih and Dong [20]. They associated to each point x in $\{0,1\}^n$ a local interaction graph $Gf(x)$, which is a subgraph of $G(f)$ defined as the directed graph whose the adjacency matrix is the discrete Jacobian matrix of f evaluated at point x, and they proved that *if $Gf(x)$ has no directed cycle for all x in $\{0,1\}^n$, then f has a unique fixed point*.

In this paper, we generalize Shih-Dong's theorem using, as main tool, the *subnetworks* of f, that is, the networks obtained from f by fixing to 0 or 1 some components. The organization is the following. After introducing the main concepts in Section 2, we formally state some classical results connected to this work, as Robert's and Shih-Dong's theorems. In Section 4, we define the class $\mathcal{F}$ of even and odd-self-dual networks, and we prove the main result of this paper, the following characterization: *f and all its subnetworks have a unique fixed point if and only if f has no subnetworks in $\mathcal{F}$*. The rest of the paper discusses this "forbidden subnetworks theorem". In section 5, we show that it generalizes Shih-Dong's theorem. More precisely, we show how it can be used to replace the condition "$Gf(x)$ has no cycles for all x" in Shih-Dong's theorem by a weaker condition of the form "$Gf(x)$ has short cycles for few points x". In section 6, we study the effect of the absence of subnetwork in $\mathcal{F}$ on the asynchronous state graph of f (which is a directed graph on $\{0,1\}^n$ constructed from the asynchronous iterations of f and proposed by Thomas [22] as a model for the dynamics of gene networks). Section 7 gives some reflexions on the characterization of properties by forbidden subnetworks. In particular, it is showed that there is not a lot of properties that are interesting to characterize in terms of forbidden subnetworks. In Section 8, we compare $\mathcal{F}$ with the with the classes $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) of networks f such that the interaction graph $G(f)$ is a positive (resp. negative) cycle. We show that $\mathcal{C}^+$ (resp. $\mathcal{C}^-$) contains exactly the non-expansive even-self-dual (resp. odd-self-dual) networks, in such a way that $\mathcal{C}^+ \cup \mathcal{C}^-$ equals the non-expansive networks of $\mathcal{F}$. This result is used in Section 9 to obtain a strong version of the main result for non-expansive networks: *If f is non-expansive, then f and all its subnetworks have at least (resp. at most) one fixed point if and only if f has no subnetworks in $\mathcal{C}^-$ (resp. $\mathcal{C}^+$)*. In Section 10, we focus on conjunctive networks. We prove that *if f is a conjunctive network, then f and all its subnetworks have at most one (resp. a unique) fixed point if and only if f has no subnetworks in $\mathcal{C}^+$ (resp. $\mathcal{C}^+ \cup \mathcal{C}^-$)*. Finally, we show that, for conjunctive networks, the absence of subnetwork in $\mathcal{C}^\pm$ can be easily verified from the chordless cycles of $G(f)$.

2. Preliminaries

2.1. Notations on hypercube

If A and B are two sets, then A^B denotes the set of functions from B to A. Let $\mathbb{B} = \{0, 1\}$ and let V be a finite set. Elements of $\mathbb{B}^V$ are seen as *points* of the $|V|$-dimensional Boolean space, and the elements of V as the *components* (or dimensions) of this space. Given a point $x \in \mathbb{B}^V$ and a component $i \in V$, the image of i by x (the i-component of x) is denoted x_i or $(x)_i$. The set of components i such that $x_i = 1$ is denoted $\mathbf{1}(x)$. For all $I \subseteq V$, we denote by e_I the point of $\mathbb{B}^V$ such that $\mathbf{1}(e_I) = I$. Points $e_\emptyset$ and e_V are often denoted 0 and 1, and we write e_i instead of $e_{\{i\}}$. Hence, e_i may be seen as the base vector of $\mathbb{B}^V$ associated with dimension i. For all $x \in \mathbb{B}^V$, we set $\|x\| = |\mathbf{1}(x)|$. A point x is said to be even (resp. odd) if $\|x\|$ is even (resp. odd). The sum modulo two is denoted $\oplus$. If x and y are two points of $\mathbb{B}^V$, then $x \oplus y$ is the point of $\mathbb{B}^V$ such that $(x \oplus y)_i = x_i \oplus y_i$ for all $i \in V$. The *Hamming distance* between x and y is $d(x, y) = \|x \oplus y\|$. Thus $d(x, y)$ is the number of components i such that $x_i \neq y_i$. In this way $\|x\| = d(0, x)$. For all $I \subseteq V$ and $x \in \mathbb{B}^V$, the restriction of x to I is denoted $x|_I$, and the restriction of x to $V \setminus I$ is denoted x_{-I}. If $i \in V$, we write $x|_i$ and x_{-i} instead of $x|_{\{i\}}$ and $x_{-\{i\}}$. Also, if $\alpha \in \mathbb{B}$ then $x^{i\alpha}$ denotes the point of $\mathbb{B}^V$ such that $(x^{i\alpha})_i = \alpha$ and $(x^{i\alpha})_{-i} = x_{-i}$.

2.2. Networks and subnetworks

A **(Boolean) network** on V is a function $f : \mathbb{B}^V \to \mathbb{B}^V$. The elements of V are the *components* or *automata* of the network, and $\mathbb{B}^V$ is the set of possible *states* or *configurations* for the network. At a given configuration $x \in \mathbb{B}^V$, the state of component i is given by x_i. The *local transition function* associated with component i is the function f_i from $\mathbb{B}^V$ to $\mathbb{B}$ defined by $f_i(x) = f(x)_i$ for all $x \in \mathbb{B}^V$. Throughout this article, f denotes a network on V.

We say that f is **non-expansive** if

$$\forall x, y \in \mathbb{B}^V, \qquad d(f(x), f(y)) \leq d(x, y).$$

The **conjugate** of f is the network $\tilde{f}$ on V defined by

$$\forall x \in \mathbb{B}^V, \qquad \tilde{f}(x) = f(x) \oplus x.$$

Let I be a non-empty subset of V and $z \in \mathbb{B}^{V \setminus I}$. The **subnetwork** of f induced by z is the network h on I defined by

$$\forall x \in \mathbb{B}^V \text{ with } x_{-I} = z, \qquad h(x|_I) = f(x)|_I.$$

The subnetwork of f induced by z is thus the network obtained from f by fixing to z_i each component $i \in V \setminus I$. It can also be seen as the projection of the restriction of f to the hyperplane defined by the equations "$x_i = z_i$", $i \in V \setminus I$. Note that, by definition, f *is a subnetwork of itself.* A subnetwork of

f distinct from f is a **strict subnetwork**. Let $i \in V$, $\alpha \in \mathbb{B}$ and let $z \in \mathbb{B}^V$ with $z_i = \alpha$. The subnetwork of f induced by $z|_i$ is denoted $f^{i\alpha}$ and called **immediate subnetwork** of f induced by the hyperplane "$x_i = \alpha$". In other words,

$$\forall x \in \mathbb{B}^V, \qquad f^{i\alpha}(x_{-i}) = f(x^{i\alpha})_{-i}.$$

2.3. Asynchronous state graph

The **asynchronous state graph** of f, denoted $\Gamma(f)$, is the directed graph with vertex set $\mathbb{B}^V$ and the following set of arcs:

$$\{x \to x \oplus e_i \,|\, x \in \mathbb{B}^V,\ i \in V,\ f_i(x) \neq x_i\}$$

Remark 1. Our interest for $\Gamma(f)$ lies in the fact that this state graph has been proposed by Thomas [22] as a model for the dynamics of gene networks; see also [24]. In this context, network components correspond to genes. At a given state x, the protein encoded by gene i is "present" if $x_i = 1$ and "absent" if $x_i = 0$. The gene i is "on" (transcripted) if $f_i(x) = 1$ and "off" (not transcripted) if $f_i(x) = 0$. And given an initial configuration x, the possible evolutions of the system are described by the set of paths of $\Gamma(f)$ starting from x.

The terminal strongly connected components of $\Gamma(f)$ are called **attractors**. An attractor is **cyclic** if it contains at least two points, and it is **punctual** otherwise. Hence, $\{x\}$ is a punctual attractor of $\Gamma(f)$ if and only if x is a fixed point of f, so both concepts are identical.

Proposition 1. *Let I be non-empty subset of V and let h be the subnetwork of f induced by some point $z \in \mathbb{B}^{V\setminus I}$. The asynchronous state graph of h is isomorphic to the asynchronous state graph of f induced by the set of points $x \in \mathbb{B}^V$ such that $x_{-I} = z$ (the isomorphism is $x \mapsto x_{-I}$).*

Proof. For all $x, y \in \mathbb{B}^V$ with $x_{-I} = y_{-I} = z$, and for all $i \in I$, we have $y = x \oplus e_i$ if and only if $x|_I = y|_I \oplus e_i$, thus $x \to y$ is an arc of $\Gamma(f)$ if and only if $x_{-I} \to y_{-I}$ is an arc of $\Gamma(h)$. □

2.4. Criticality

We say that f is **critical** for a property $\mathcal{P}$, if f has the property $\mathcal{P}$ but no strict subnetworks of f have this property. Let $\mathcal{P}_2$ be the property "to have at least two fixed points", and let $\mathcal{P}_0$ be the property "to have no fixed point". We say that f is **2-critical** if f is critical for the property $\mathcal{P}_2$, and we say that f is **0-critical** if f is critical for the property $\mathcal{P}_0$. Clearly, if f is 2-critical, then there exists $x \in \mathbb{B}^V$ such that x and $x \oplus 1$ are fixed points, and f has no other fixed point (because if x and y are two fixed points and $x_i = y_i = \alpha$ then x_{-i} and y_{-i} are fixed points of $f^{i\alpha}$).

Proposition 2. *Let f be a network on V.*

1. *If the asynchronous state graph of f has multiple attractors, then f has a 2-critical subnetwork.*

2. *If f is non-expansive and if the asynchronous state graph of f has a cyclic attractor, then f has no fixed point and thus has a 0-critical subnetwork.*

Proof. Suppose that $\Gamma(f)$ has two distinct attractors $X, Y \subseteq \mathbb{B}^V$. Let $x \in X$ and $y \in Y$ be such that $d(x, y)$ is minimal. Let $I = \mathbf{1}(x \oplus y)$ so that $x_{-I} = y_{-I} = z$. Let h be the subnetwork of f induced by z. Suppose that $x|_I$ is not a fixed point of h. Then, there exists $i \in I$ with $x_i \neq h_i(x|_I) = f_i(x)$. Thus $\Gamma(f)$ has an arc $x \to x \oplus e_i$ and $x \oplus e_i \in X$ because $x \in X$. Since $x_i \neq y_i$, we have $d(x \oplus e_i, y) < d(x, y)$, a contradiction. Thus $x|_I$ is a fixed point of h, and we prove with similar arguments that $y|_I$ is a fixed point of h. Thus h has multiple fixed points. Thus h has necessarily a 2-critical subnetwork g, and since g is a subnetwork of f the first point is proved.

For the second point, suppose in addition that f is non-expansive, that Y is a cyclic attractor (*i.e.* $|Y| > 1$) and that X is punctual *i.e.* reduces to a fixed point x of f. Since $y|_I$ is a fixed point of h, we have $y|_I = f(y)|_I$ and using the fact that f is non expansive we get

$$\begin{aligned} d(f(x), f(y)) = d(x, f(y)) = d(x|_I, f(y)|_I) + d(x_{-I}, f(y)_{-I}) = \\ d(x|_I, y|_I) + d(y_{-I}, f(y)_{-I}) = |I| + d(y_{-I}, f(y)_{-I}) \leq d(x, y) = |I|. \end{aligned}$$

Thus $d(y_{-I}, f(y)_{-I}) = 0$ so $y_{-I} = f(y)_{-I}$. Consequently, y is a fixed point of f, and Y cannot be cyclic, a contradiction. Consequently, if f is non-expansive and if Y is a cyclic attractor, then X is also a cyclic attractor, so f has no fixed point and thus it has necessarily a 0-critical subnetwork. □

2.5. Interaction graphs

Notions and notations concerning digraphs are consistent with [3]. In particular, cycles and paths are seen as digraphs and thus have no repeated vertices. A **signed digraph** $G = (V, A)$ consists in a set of vertices V and a set of (signed) arcs $A \subseteq V \times \{-1, 1\} \times V$. An arc $(i, s, j) \in A$ is an arc from i to j of sign s. We say that G is **simple** if for every vertices $i, j \in V$ there is at most one arc from i to j. The (unsigned) digraph obtained by forgetting signs is denoted $|G|$: The vertex set of $|G|$ is V and the arc set of $|G|$ is the set of couples (i, j) such that G has at least one arc from i to j. A signed digraph $G' = (V', A')$ is a subgraph of G (notation $G' \subseteq G$) if $V' \subseteq V$ and $A' \subseteq A$. A **cycle** of G is a simple subgraph C of G such that $|C|$ is a directed cycle. A **positive** (resp. **negative**) **cycle** of G is a cycle of G with an even (resp. odd) number of negative arcs. A cycle of C of G is **chordless** if $|C|$ is an induced subgraph of $|G|$ (*i.e.* $|C|$ can be obtained from $|G|$ be removing vertices only).

Let f be a network on V and two components $i, j \in V$. The **discrete derivative** of f_i with respect to j is the function f_{ij} from $\mathbb{B}^V$ to $\{-1, 0, 1\}$ defined by

$$\forall x \in \mathbb{B}^V, \qquad f_{ij}(x) = f_i(x^{j1}) - f_i(x^{j0}).$$

Discrete derivatives are usually stored under the form of a matrix, the *Jacobian matrix*. However, for our purpose, it is more convenient to store them under the form of a signed digraph.

For all $x \in \mathbb{B}^V$, we call **local interaction graph** of f evaluated at point x, and we denote by $Gf(x)$, the signed digraph with vertex set V such that, for all $i, j \in V$, there is a positive (resp. negative) arc from j to i if $f_{ij}(x)$ positive (resp. negative). Note that $Gf(x)$ is simple. The **(global) interaction graph** of f is the signed digraph denoted by $G(f)$ and defined by: The vertex set is V and, for all vertices $i, j \in V$, there is a positive (resp. negative) arc from j to i if $f_{ij}(x)$ is positive (resp. negative) for at least one $x \in \mathbb{B}^V$. Thus each local interaction graph $Gf(x)$ is a subgraph of the global interaction graph $G(f)$. More precisely, $G(f)$ is obtained by taking the union of all the $Gf(x)$.

Proposition 3. *Let I be non-empty subset of V and let h be the subnetwork of f induced by some point $z \in \mathbb{B}^{V \setminus I}$, and let $x \in \mathbb{B}^V$ with $x_{-I} = z$. Then:*

1. *$Gh(x|_I)$ is an induced subgraph of $Gf(x)$;*
2. *$G(h)$ is a subgraph of $G(f)$.*

Proof. If $x \in \mathbb{B}^V$ and $x_{-I} = z$, then for all $i, j \in I$,

$$h_{ij}(x|_I) = h_i(x^{j1}|_I) - h_i(x^{j0}|_I) = f_i(x^{j1}) - f_i(x^{j0}) = f_{ij}(x).$$

This proves *1.* and *2.* is an obvious consequence. □

3. Some fixed point theorems

Robert proved in 1980 the following fundamental fixed point theorem [16, 17]. A short proof is given in Appendix A (this proof uses an induction on subnetworks, a technic used in almost all proofs of this paper).

Theorem 1 (Robert 1980)**.** *If $G(f)$ has no cycle then f has a unique fixed point.*

Robert also proved, in his french book [18], that if $G(f)$ has no cycle, then $\Gamma(f)$ has no cycle, so that every path of $\Gamma(f)$ leads to the unique fixed point of f (strong convergence toward a unique fixed point).

The following theorem, proved by Aracena [1] (see also [2]) in a slightly different setting, gives other very fundamental relationships between the interaction graph of f and its fixed points.

Theorem 2 (Aracena 2008)**.** *Suppose that $G(f)$ is strongly connected (and contains at least one arc).*

1. *If $G(f)$ has no negative cycle then f has at least two fixed points.*
2. *If $G(f)$ has no positive cycle then f has no fixed point.*

The following theorem can be deduce from Aracena theorem with an induction on strongly connected components of $G(f)$, see Appendix A. It gives a nice "proof by dichotomy" of Robert' theorem: The *existence* of a fixed point is established under the absence of *negative cycle* while the *unicity* under the absence of *positive cycle*.

Theorem 3.

1. *If $G(f)$ has no positive cycle then f has at most one fixed point.*
2. *If $G(f)$ has no negative cycle then f has at least one fixed point.*

First point of Theorem 3 can be seen as a Boolean version of first Thomas' rule, which asserts that the presence of a positive cycles in the interaction graph of a dynamical system is a necessary conditions for the presence of multiple stable states [23] (see also [8] and the references therein).

Second Thomas' rule asserts that the presence of a negative cycle is a necessary condition for the presence of cyclic attractors [23, 8]. Hence, the next theorem, proved in [12], can be see as a Boolean version of second Thomas' rule.

Theorem 4 (Richard 2010)**.** *If $G(f)$ has no negative cycle, then $\Gamma(f)$ has no cyclic attractors.*

Note that this theorem generalizes the second point of Theorem 3: If $\Gamma(f)$ has no cyclic attractor, then all the attractors are fixed points, and since there always exists at least one attractor, f has at least one fixed point.

The next theorem is a "local version" of Robert's theorem. It has been conjectured and presented as a combinatorial analog of the Jacobian conjecture in [21]. It has be proved by Shih and Dong in [20].

Theorem 5 (Shih and Dong 2005)**.** *If $Gf(x)$ has no cycle for all $x \in \mathbb{B}^V$, then f has a unique fixe point.*

This theorem generalizes Robert's one: If $G(f)$ has no cycle, then it is clear that each local interaction graph $Gf(x)$ has no cycle (because $Gf(x) \subseteq G(f)$). The original proof of Shih and Dong is quite involved. A much more simple proof is given in Appendix A.

In a similar way, Remy, Ruet and Thieffry [11] proved a local version of the first point of Theorem 3. They thus got the uniqueness part of Shih-Dong's theorem under weaker conditions.

Theorem 6 (Remy, Ruet and Thieffry 2008)**.** *If $Gf(x)$ has no positive cycle for all $x \in \mathbb{B}^V$, then f has at most one fixed point.*

In view of the previous theorem, it very natural think about a local version of the second point of Theorem 3.

Question 1. *Is it true that if $Gf(x)$ has no negative cycle for all $x \in \mathbb{B}^V$, then f has at least one fixed point?*

The following theorem, proved in [13], only gives a very partial answer to this question (see [15] for another very partial answer).

Theorem 7 (Richard 2011)**.** *If f is non-expansive and if $Gf(x)$ has no negative cycle for all $x \in \mathbb{B}^V$, then f has at least one fixed point.*

Remark 2. In all the theorems, Aracena one excepted, if the conditions are satisfied by f then they are also satisfied by every subnetwork of f, in such a way that conclusions apply to f and all its subnetworks. For instance, if $G(f)$ has no cycle, then the interaction graph $G(h)$ every subnetwork h of f has no cycle (since $G(h) \subseteq G(f)$), and by Robert's theorem, every subnetwork h of f has a unique fixed point. Such a remark is also valid for Theorem 7, because if f is non-expansive then all its subnetworks are non-expansive too.

Remark 3. Using the previous remark, we deduce from Theorem 5 (resp. Theorem 6) that if $Gf(x)$ has no cycle (resp. no positive cycle) for all $x \in \mathbb{B}^V$, then every subnetwork of f has a unique (resp. at most one) fixed point, and thus, following Proposition 2, $\Gamma(f)$ has a unique attractor (resp. at most one attractor).

Remark 4. Proceeding in a similar way, we deduce from Theorem 7 and Proposition 2 the following local version of second Thomas' rule for non-expansive networks: *If f is non-expansive and if $Gf(x)$ has no negative cycle for all $x \in \mathbb{B}^V$, then $\Gamma(f)$ has no cyclic attractors.*

4. A forbidden subnetwork theorem

In this section, we introduce the class $\mathcal{F}$ of even- and odd-self dual networks, and we prove that it has the following property: Every subnetworks of f (and f itself in particular) has a unique fixed point if and only if f has no subnetwork in $\mathcal{F}$.

We say that f is **self-dual** if

$$\forall x \in \mathbb{B}^V, \qquad f(x \oplus 1) = f(x) \oplus 1.$$

Equivalently, f is self-dual if $\tilde{f}(x \oplus 1) = \tilde{f}(x)$ for all $x \in \mathbb{B}^V$.

We say that f is **even** if the image set of $\tilde{f}$ is the set of even points of $\mathbb{B}^V$, that is,

$$\tilde{f}(\mathbb{B}^V) = \{x \in \mathbb{B}^V \mid \|x\| \text{ is even}\}$$

and similarly, we say that f is **odd** if

$$\tilde{f}(\mathbb{B}^V) = \{x \in \mathbb{B}^V \mid \|x\| \text{ is odd}\}.$$

Thus, if f is even, then there exists $x \in \mathbb{B}^V$ such that $\tilde{f}(x) = 0$, which is equivalent to say that $f(x) = x$. Hence, even networks have at least one fixed point. Obviously, odd networks have no fixed point.

We say that f is **even-self-dual** (resp. **odd-self-dual**) if it is both even (resp. odd) and self dual. We will often implicitly use the following characterization: f is even-self-dual (resp. odd-self-dual) if and only if

$$\forall z \in \mathbb{B}^V \text{ s.t. } \|z\| \text{ is even (resp. odd)},\ \exists x \in \mathbb{B}^V \text{ s.t. } \tilde{f}^{-1}(z) = \{x, x \oplus 1\}.$$

It follows that if f is even-self-dual then it has exactly two fixed points.

Our interest for even- or odd-self-dual networks lies in the following theorem, which is the main result of this paper.

Theorem 8. *If f has no even- or odd-self-dual subnetwork, then the conjugate of f is a bijection.*

The proof needs the following two lemmas.

Lemma 1. *Let X be a non-empty subset of $\mathbb{B}^V$ and*

$$N(X) = \{x \oplus e_i \,|\, x \in X,\, i \in V\}.$$

If X and $N(X)$ are disjoint and $|X| \geq |N(X)|$, then X is either the set of even points of $\mathbb{B}^V$ or the set of odd points of $\mathbb{B}^V$.

Proof. by induction on $|V|$. The case $|V| = 1$ is obvious. So suppose that $|V| > 1$. Let X be a non-empty subset of $\mathbb{B}^V$ satisfying the conditions of the statement. Let $i \in V$ and $\alpha \in \mathbb{B}$. For all $Y \subseteq \mathbb{B}^V$, let us denote by Y^α be the subset of $\mathbb{B}^{V \setminus \{i\}}$ defined by $Y^\alpha = \{x_{-i} \,|\, x \in Y, x_i = \alpha\}$.

We first prove that $N(X^\alpha) \subseteq N(X)^\alpha$ and $X^\alpha \cap N(X^\alpha) = \emptyset$. Let $x \in \mathbb{B}^V$ with $x_i = \alpha$ be such that $x_{-i} \in N(X^\alpha)$. To prove that $N(X^\alpha) \subseteq N(X)^\alpha$, it is sufficient to prove that $x_{-i} \in N(X)^\alpha$. Since $x_{-i} \in N(X^\alpha)$, there exists $y \in \mathbb{B}^V$ with $y_i = \alpha$ and $j \in V$ with $j \neq i$ such that $y_{-i} \in X^\alpha$ and $x_{-i} = y_{-i} \oplus e_j$. So $x = y \oplus e_j$, and since $y_i = \alpha$, we have $y \in X$. Hence $x \in N(X)$ and since $x_i = \alpha$, we have $x_{-i} \in N(X)^\alpha$. We now prove that $X^\alpha \cap N(X^\alpha) = \emptyset$. Indeed, otherwise, there exists $x \in \mathbb{B}^V$ with $x_i = \alpha$ such that $x_{-i} \in X^\alpha \cap N(X^\alpha)$. Since $N(X^\alpha) \subseteq N(X)^\alpha$, we have $x_{-i} \in X^\alpha \cap N(X)^\alpha$, and since $x_i = \alpha$, we deduce that $x \in X \cap N(X)$, a contradiction.

Since $N(X^\alpha) \subseteq N(X)^\alpha$, we have

$$|X| = |X^0| + |X^1| \geq |N(X)| = |N(X)^0| + |N(X)^1| \geq |N(X^0)| + |N(X^1)|.$$

So $|X^0| \geq |N(X^0)|$ or $|X^1| \geq |N(X^1)|$. Suppose that $|X^0| \geq |N(X^0)|$, the other case being similar. Since $X^0 \cap N(X^0) = \emptyset$, by induction hypothesis X^0 is either the set of even points of $\mathbb{B}^{V \setminus \{i\}}$ or the set of odd points of $\mathbb{B}^{V \setminus \{i\}}$. So in both cases, we have $|X^0| = |N(X^0)| = 2^{|V|-1}$. We deduce that $|X^1| \geq |N(X^1)|$, and so, by induction hypothesis, X^1 is either the set of even points of $\mathbb{B}^{V \setminus \{i\}}$ or the set of odd points of $\mathbb{B}^{V \setminus \{i\}}$. But X^0 and X^1 are disjointed: For all $x \in \mathbb{B}^V$, if $x_{-i} \in X^0 \cap X^1$, then x^{i0} and x^{i1} are two points of X, and $x^{i1} = x^{i0} \oplus e_i \in N(X)$, a contradiction. So if X^0 is the set of even (resp. odd) points of $\mathbb{B}^{V \setminus \{i\}}$, then X^1 is the set of odd (resp. even) points of $\mathbb{B}^{V \setminus \{i\}}$, and we deduce that X is the set of even (resp. odd) points of $\mathbb{B}^V$. □

Lemma 2. *Suppose that the conjugate of every immediate subnetwork of a network f is a bijection. If the conjugate of f is not a bijection, then f is even- or odd-self-dual.*

Proof. Suppose that $f : \mathbb{B}^V \to \mathbb{B}^V$ satisfies the conditions of the statement, and suppose that the conjugate $\tilde{f}$ of f is not a bijection. Let

$$X = \tilde{f}(\mathbb{B}^V), \qquad \bar{X} = \mathbb{B}^V \setminus X.$$

Since $\tilde{f}$ is not a bijection, $\bar{X}$ is not empty.

Let us first prove that

$$\forall x \in \bar{X},\ \forall i \in V, \qquad |\tilde{f}^{-1}(x \oplus e_i)| = 2. \tag{$*$}$$

Let $x \in \bar{X}$ and $i \in V$. By hypothesis, $\tilde{f}^{i0}$ is a bijection, so there exists a unique point $y \in \mathbb{B}^V$ with $y_i = 0$ such that $\tilde{f}^{i0}(y_{-i}) = x_{-i}$. Then, $\tilde{f}(y)_{-i} = \tilde{f}(y^{i0})_{-i} = \tilde{f}^{i0}(y_{-i}) = x_{-i}$. In other words $\tilde{f}(y) \in \{x, x \oplus e_i\}$. Since $x \in \bar{X}$ we have $\tilde{f}(y) \neq x$ and it follows that $\tilde{f}(y) = x \oplus e_i$. Hence, we have proved that there exists a unique point $y \in \mathbb{B}^V$ such that $y_i = 0$ and $\tilde{f}(y) = x \oplus e_i$, and we prove with similar arguments that there exists a unique point $z \in \mathbb{B}^V$ such that $z_i = 1$ and $\tilde{f}(z) = x \oplus e_i$. This proves $(*)$.

We are now in position to prove that f is even or odd. Let

$$N(\bar{X}) = \{x \oplus e_i \,|\, x \in \bar{X},\, i \in V\}.$$

Following $(*)$ we have $N(\bar{X}) \subseteq X$, and we deduce that

$$\begin{aligned} |\tilde{f}^{-1}(X)| &= |\tilde{f}^{-1}(N(\bar{X}))| + |\tilde{f}^{-1}(X \setminus N(\bar{X}))| \\ &\geq |\tilde{f}^{-1}(N(\bar{X}))| + |X \setminus N(\bar{X})| \\ &= |\tilde{f}^{-1}(N(\bar{X}))| + |X| - |N(\bar{X})|. \end{aligned}$$

Again following $(*)$, $|\tilde{f}^{-1}(N(\bar{X}))| = 2|N(\bar{X})|$ and we deduce that

$$|X| + |\bar{X}| = 2^{|V|} = |\tilde{f}^{-1}(X)| \geq 2|N(\bar{X})| + |X| - |N(\bar{X})| = |N(\bar{X})| + |X|.$$

Therefore, $|\bar{X}| \geq |N(\bar{X})|$, and since $N(\bar{X}) \subseteq X = \mathbb{B}^{|V|} \setminus \bar{X}$, we have $\bar{X} \cap N(\bar{X}) = \emptyset$. So according to Lemma 1, $\bar{X}$ is either the set of even points of $\mathbb{B}^{|V|}$ or the set of odd points of $\mathbb{B}^{|V|}$. We deduce that in the first (second) case, X is the set of odd (even) points of $\mathbb{B}^{|V|}$. Thus, f is even or odd.

It remains to prove that f is self-dual. Let $x \in \mathbb{B}^V$. For all $i \in V$, since $\|\tilde{f}(x)\|$ and $\|\tilde{f}(x) \oplus e_i\|$ have not the same parity, and since f is even or odd, we have $\tilde{f}(x) \oplus e_i \in \bar{X}$. Thus, according to $(*)$, the preimage of $(\tilde{f}(x) \oplus e_i) \oplus e_i = \tilde{f}(x)$ by $\tilde{f}$ is of cardinality two. Consequently, there exists a point $y \in \mathbb{B}^{|V|}$, distinct from x, such that $\tilde{f}(y) = \tilde{f}(x)$. Let us proved that $x = y \oplus 1$. Indeed, if $x_i = y_i = 0$ for some $i \in V$, then $\tilde{f}^{i0}(x_{-i}) = \tilde{f}(x)_{-i} = \tilde{f}(y)_{-i} = \tilde{f}^{i0}(y_{-i})$. Since $x \neq y$, we deduce that $\tilde{f}^{i0}$ is not a bijection, a contradiction. We show similarly that if $x_i = y_i = 1$, then $\tilde{f}^{i1}$ is not a bijection. So $x = y \oplus 1$. Consequently, $\tilde{f}(x \oplus 1) = \tilde{f}(x)$, and we deduce that f is self-dual. □

Proof of Theorem 8. by induction on $|V|$. The case $|V| = 1$ is obvious. So suppose that $|V| > 1$ and suppose that f has no even- or odd-self-dual subnetwork. Under this condition, f is neither even-self-dual nor odd-self-dual (since f is a subnetwork of f), and every immediate subnetwork of f has no even- or odd-self-dual subnetwork. So, by induction hypothesis, the dual of every strict subnetwork of f is a bijection, and we deduce from Lemma 2 that the dual of f is a bijection. □

Corollary 1. *The conjugate of each subnetwork of f is a bijection if and only if f has no even- or odd-self-dual subnetworks.*

Proof. If f has no even- or odd-self-dual subnetwork, then every subnetwork h of f has no even- or odd-self-dual subnetwork, and according to Theorem 8, the conjugate of h is a bijection. Conversely, if the conjugate of each subnetwork of f is a bijection, then f has clearly no even- or odd-self-dual subnetwork (since if a network is even or odd, its conjugate sends $\mathbb{B}^V$ to a subset of $\mathbb{B}^V$ of cardinality $|\mathbb{B}^V|/2$). □

If $\tilde{f}$ is a bijection then there is a unique point $x \in \mathbb{B}^V$ such that $\tilde{f}(x) = 0$, and this point is thus the unique fixed point of f. As an immediate consequence of this property and the previous corollary, we obtain the characterization mentioned at the beginning of the section.

Corollary 2. *Each subnetwork of f has a unique fixed point (f in particular) if and only if f has no even- or odd-self-dual subnetworks.*

Remark 5. As an immediate consequence of the two previous corollary, we get the following property, independently proved by Ruet in [19]: *Each subnetwork of f has a unique fixed point if and only if the conjugate of each subnetwork of f is a bijection.*

Example 1. Consider the following network f on $\{1, 2, 3\}$ [1]:

$$f : \mathbb{B}^{\{1,2,3\}} \to \mathbb{B}^{\{1,2,3\}} \qquad \begin{aligned} f_1(x) &= \overline{x_2} \wedge x_3 \\ f_2(x) &= \overline{x_3} \wedge x_1 \\ f_3(x) &= \overline{x_1} \wedge x_2. \end{aligned}$$

[1] In all the examples, network components are integers, and if V is a set of n integers $i_1 < i_2 < \cdots < i_n$, then for all $x \in \mathbb{B}^V$ we write $x = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ or $x = x_{i_1} x_{i_2} \ldots x_{i_n}$.

The table of f and $\tilde{f}$ are:

x	$f(x)$	$\tilde{f}(x)$
000	000	000
001	100	101
010	001	011
011	001	010
100	010	110
101	100	001
110	010	100
111	000	111

The six immediate subnetworks of f are:

$$f^{10} : \mathbb{B}^{\{2,3\}} \to \mathbb{B}^{\{2,3\}} \qquad \begin{array}{l} f_2^{10}(x) = \overline{x_3} \wedge 0 = 0 \\ f_3^{10}(x) = \overline{0} \wedge x_2 = x_2 \end{array}$$

$$f^{11} : \mathbb{B}^{\{2,3\}} \to \mathbb{B}^{\{2,3\}} \qquad \begin{array}{l} f_2^{11}(x) = \overline{x_3} \wedge 1 = \overline{x_3} \\ f_3^{11}(x) = \overline{1} \wedge x_2 = 0 \end{array}$$

$$f^{20} : \mathbb{B}^{\{1,3\}} \to \mathbb{B}^{\{1,3\}} \qquad \begin{array}{l} f_1^{20}(x) = \overline{0} \wedge x_3 = x_3 \\ f_3^{20}(x) = \overline{x_1} \wedge 0 = 0 \end{array}$$

$$f^{21} : \mathbb{B}^{\{1,3\}} \to \mathbb{B}^{\{1,3\}} \qquad \begin{array}{l} f_1^{21}(x) = \overline{1} \wedge x_3 = 0 \\ f_3^{21}(x) = \overline{x_1} \wedge 1 = \overline{x_1} \end{array}$$

$$f^{30} : \mathbb{B}^{\{1,2\}} \to \mathbb{B}^{\{1,2\}} \qquad \begin{array}{l} f_1^{30}(x) = \overline{x_2} \wedge 0 = 0 \\ f_2^{30}(x) = \overline{0} \wedge x_1 = x_1 \end{array}$$

$$f^{31} : \mathbb{B}^{\{1,2\}} \to \mathbb{B}^{\{1,2\}} \qquad \begin{array}{l} f_1^{31}(x) = \overline{x_2} \wedge 1 = \overline{x_2} \\ f_2^{31}(x) = \overline{1} \wedge x_1 = 0 \end{array}$$

So each immediate subnetwork $f^{i\alpha}$ of f has one component fixed to zero, so f has no self-dual immediate subnetwork. Furthermore, each immediate subnetwork of $f^{i\alpha}$ is a constant (0), and thus is not self-dual. Furthermore, f is not self-dual since $f(000) = f(111) = 111$. Hence, f has no self-dual subnetwork, and we deduce from Theorem 8 that the conjugate of $\tilde{f}$ of f is a bijection. This can be easily verified on the table given above.

5. Generalization of Shih-Dong's theorem

In this section, we show, using Theorem 8, that the condition "$Gf(x)$ has no cycles for all x" in Shih-Dong's theorem (Theorem 5) can be weakened into a condition of the form "$Gf(x)$ has short cycles for few points x". The exact statement is given after the following useful proposition.

Proposition 4. *If f is even or odd, then for every $x \in \mathbb{B}^V$ the out-degree of each vertex of $Gf(x)$ is odd. In particular, $Gf(x)$ has a cycle.*

Proof. Let $j \in V$ and let d be the out-degree of j in $Gf(x)$. Since d equals the number of $i \in V$ such that $|f_{ij}(x)| = 1$, and since

$$|f_{ij}(x)| = f_i(x^{j1}) \oplus f_i(x^{j0}) = f_i(x) \oplus f_i(x \oplus e_j),$$

we have

$$\begin{aligned} d &= \|f(x) \oplus f(x \oplus e_j)\| \\ &= \|(x \oplus \tilde{f}(x)) \oplus ((x \oplus e_j) \oplus \tilde{f}(x \oplus e_j))\| \\ &= \|\tilde{f}(x) \oplus \tilde{f}(x \oplus e_j) \oplus e_j\|. \end{aligned}$$

So the parity of d is the parity of $\|\tilde{f}(x)\| + \|\tilde{f}(x \oplus e_i)\| + 1$. Hence, if f is even or odd, then $\|\tilde{f}(x)\|$ and $\|\tilde{f}(x \oplus e_i)\|$ have the same parity, so $\|\tilde{f}(x)\| + \|\tilde{f}(x \oplus e_i)\|$ is even, and it follows that d is odd. □

Corollary 3. *If, for every $1 \leq k \leq |V|$, there exists at most $2^k - 1$ points $x \in \mathbb{B}^V$ such that $Gf(x)$ has a cycle of length at most k, then f has a unique fixed point.*

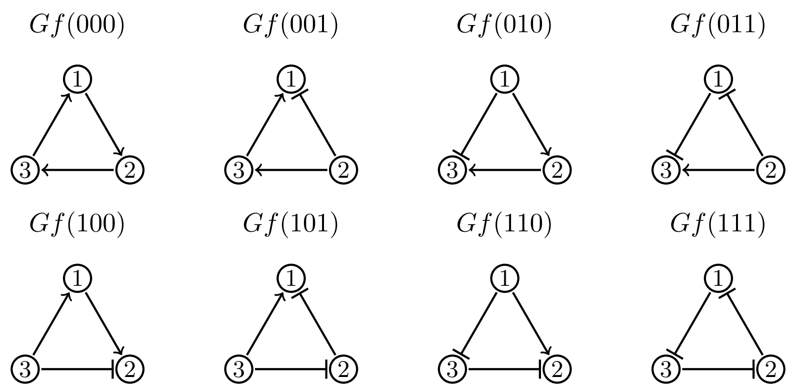
Proof. According to Theorem 8, it is sufficient to prove, by induction on $|V|$, that if f satisfies the conditions of the statement, then f has no even- or odd-self-dual subnetwork. The case $|V| = 1$ is obvious, so suppose that $|V| > 1$. Suppose also that f satisfies the conditions of the statement. Let $i \in V$ and $\alpha \in \mathbb{B}$. Since $Gf^{i\alpha}(x_{-i})$ is the subgraph of $Gf(x^{i\alpha})$ for all $x \in \mathbb{B}^V$ (cf. Proposition 3), $f^{i\alpha}$ satisfies the condition of the theorem. Thus, by induction hypothesis, $f^{i\alpha}$ has no even- or odd-self-dual subnetwork. So f has no even- or odd-self-dual strict subnetwork. If f is itself even- or odd-self-dual, then by Proposition 4, $Gf(x)$ has a cycle for every $x \in \mathbb{B}^V$, so f does not satisfy that conditions of the statement (for $k = |V|$). Therefore, f has no even- or odd-self-dual subnetwork. □

Example 2. [Continuation of Example 1] Take again the 3-dimensional network f defined by

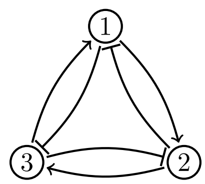
$$\begin{aligned} f_1(x) &= \overline{x_2} \wedge x_3 \\ f_2(x) &= \overline{x_3} \wedge x_1 \\ f_3(x) &= \overline{x_1} \wedge x_2. \end{aligned}$$

We have seen that f has no self-dual subnetwork. So it satisfies the conditions of Theorem 8, but not the conditions of Shih-Dong's theorem. Indeed, $Gf(000)$

and $Gf(111)$ have a cycle [2]:



However, f satisfies the condition of Corollary 3 (there is $0 < 2^1$ point x such that $Gf(x)$ has a cycle of length at most 1, $0 < 2^2$ point x such that $Gf(x)$ has a cycle of length at most 2, and $2 < 2^3$ points x such that $Gf(x)$ has a cycle of length at most 3). From the local interactions graphs given above, we deduce that the global interaction graph $G(f)$ of the network is the following:



In addition to Proposition 4, we have the following property on the structure of the interactions of even- and odd-self-dual networks.

Proposition 5. *If f is a critical even- or odd-self-dual network then $G(f)$ is strongly connected.*

Proof. Suppose that f is critical even- or odd-self-dual. If $G(f)$ is not strongly connected, then it has an initial strongly connected component I (no arc from $V \setminus I$ to I) strictly included in V. Let h be the subnetwork of f induced by some point $z \in \mathbb{B}^I$. Since f is critical and since h is a strict subnetwork, according to Theorem 8, $\tilde{h}$ is a bijection. Thus there exists $x, y \in \mathbb{B}^V$ with $x|_I = y|_I = z$, such that $\tilde{h}(x_{-I})$ and $\tilde{h}(y_{-I})$ have not the same parity. Since $\tilde{f}(x)_{-I} = \tilde{h}(x_{-I})$ and $\tilde{f}(y)_{-I} = \tilde{h}(y_{-I})$ we have

$$\|\tilde{f}(x)\| = \|\tilde{f}(x)|_I\| + \|\tilde{h}(x_{-I})\|, \qquad \|\tilde{f}(y)\| = \|\tilde{f}(y)|_I\| + \|\tilde{h}(y_{-I})\|.$$

Since $G(f)$ has no arc from $V \setminus I$ to I, and since $x|_I = y|_I$ we have $f(x)|_I = f(y)|_I$ and thus $\tilde{f}(x)|_I = \tilde{f}(y)|_I$. Thus $\tilde{f}(x)$ and $\tilde{f}(y)$ have not the same parity, a contradiction. □

[2] Arrows correspond to positive arcs and bars to negative arcs.

6. Weak asynchronous convergence

We say that the asynchronous state graph $\Gamma(f)$ describes a **strong asynchronous convergence** toward a unique fixed x if $\Gamma(f)$ is acyclic and admits x as unique attractor. We say that $\Gamma(f)$ describes a **weak asynchronous convergence** toward a unique fixed point x if $\Gamma(f)$ admits x as unique attractor (equivalently, f has a unique fixed point x and $\Gamma(f)$ has a path from any point y to x). The following corollary shows that the absence of even- or odd-self-dual subnetwork implies a weak asynchronous convergence toward a unique fixed point.

Corollary 4. *If f has no even or odd self-dual subnetwork, then f has a unique fixed point x, and for all $y \in \mathbb{B}^V$, the asynchronous state graph of f contains a path from y to x of length $d(x, y)$.*

Remark 6. By definition, if $x \to y$ is an arc of the asynchronous state graph, then $d(x, y) = 1$. Hence, path from a point x to a point y cannot be of length strictly less than $d(x, y)$; a path from x to y of length $d(x, y)$ can thus be seen as a *shortest* or *straight* path.

Proof of Corollary 4. By induction on $|V|$. The case $|V| = 1$ is obvious, so suppose that $|V| > 1$ and that f has no even or odd self-dual subnetwork. By Theorem 8, f has a unique fixed point x. Let $y \in \mathbb{B}^V$. Suppose first that there exists $i \in V$ such that $x_i = y_i = 0$. Then x_{-i} is the unique fixed point of f^{i0}. So, by induction hypothesis, $\Gamma(f^{i0})$ has a path from y_{-i} to x_{-i} of length $d(x_{-i}, y_{-i})$. Since $x_i = y_i = 0$, we deduce from Proposition 1 that $\Gamma(f)$ has a path from y to x of length $d(x_{-i}, y_{-i}) = d(x, y)$. The case $x_i = y_i = 1$ is similar. So, finally, suppose that $y = x \oplus 1$. Since y is not a fixed point, there exists $i \in V$ such that $f_i(y) \neq y_i$. Then, $\Gamma(f)$ has an arc from y to $z = y \oplus e_i$. So $z_i = x_i$, and as previously, we deduce that $\Gamma(f)$ has a path from z to x of length $d(x, z)$. This path together with the arc $y \to z$ forms a path from y to x of length $d(x, z) + 1 = d(x, y)$. □

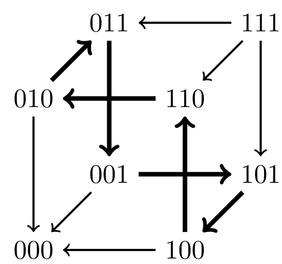
Remark 7. According to Proposition 1, the asynchronous state graph $\Gamma(h)$ of each subnetwork h of f is a subgraph of $\Gamma(f)$ induced by some subcube of $\mathbb{B}^V$. Hence, one can see $\Gamma(h)$ as a "*dynamical module*" of $\Gamma(f)$. An interpretation of the previous corollary is then that *the asynchronous state graphs of even- and odd-self-dual networks are "dynamical modules" that are necessary for the "emergence" of "complex" asynchronous behaviors*, because in their absence the dynamics is "simple": weak asynchronous convergence toward a unique fixed point.

Example 3. [Continuation of Example 1] Take again the 3-dimensional network f defined in Example 1, which has no self-dual subnetwork.

$$\begin{aligned} f_1(x) &= \overline{x_2} \wedge x_3 \\ f_2(x) &= \overline{x_3} \wedge x_1 \\ f_3(x) &= \overline{x_1} \wedge x_2. \end{aligned}$$

The asynchronous state graph $\Gamma(f)$ of f is the following:

x	$f(x)$	$\tilde{f}(x)$
000	000	000
001	100	101
010	001	011
011	001	010
100	010	110
101	100	001
110	010	100
111	000	111



In agreement with Corollary 4, there exists, from any initial point, a shortest path leading to the unique fixed point of f (the point 000): the asynchronous state graph describes a weak asynchronous convergence (by shortest paths) toward a unique fixed point. However, $\Gamma(f)$ has a cycle (of length 6), so every path does not lead to the unique fixed point: the condition "has no even or odd self-dual subnetworks" does no ensure a strong asynchronous convergence toward a unique fixed point.

7. Characterization by forbidden subnetworks

In this section, we are interested in characterizing networks properties by forbidden subnetworks, such as the characterization given by Corollary 2. We see a network property $\mathcal{P}$ as a set of networks, and given a set of networks $\mathcal{F}$, we say that $\mathcal{F}$ is a **set of forbidden subnetworks for** $\mathcal{P}$ if

$$f \in \mathcal{P} \iff \textsc{sub}(f) \cap \mathcal{F} = \emptyset,$$

where $\textsc{sub}(f)$ denotes the set of subnetworks of f. Thus, if $\mathcal{F}$ is a set of forbidden subnetworks for $\mathcal{P}$ then $\mathcal{F} \cap \mathcal{P} = \emptyset$ and $\mathcal{P}$ is **closed** for the subnetwork relation *i.e.* if $f \in \mathcal{P}$ then $\textsc{sub}(f) \subseteq \mathcal{P}$. The negation (or complement) of $\mathcal{P}$ is denoted $\neg\mathcal{P}$.

Proposition 6. *Let $\mathcal{P}$ be a set of networks closed for the subnetwork relation. There exists a unique smallest set $\mathcal{F}$ of forbidden subnetworks for $\mathcal{P}$. This set $\mathcal{F}$ is the set of networks critical for $\neg\mathcal{P}$.*

Proof. If $f \notin \mathcal{P}$, then f necessarily contains a subnetwork $h \notin \mathcal{P}$ such that $\textsc{sub}(h) \setminus h \subseteq \mathcal{P}$ *i.e.* a subnetwork critical for $\neg\mathcal{P}$. Conversely, if $f \in \mathcal{P}$ then $\textsc{sub}(f) \subseteq \mathcal{P}$ and since networks critical for $\neg\mathcal{P}$ are in $\neg\mathcal{P}$, f has no subnetworks critical for $\neg\mathcal{P}$. This proves that the set of networks critical for $\neg\mathcal{P}$ is a set of forbidden subnetworks for $\mathcal{P}$.

Now, suppose that $\mathcal{F}$ is a set of forbidden subnetworks for $\mathcal{P}$, let f be any network critical for $\neg\mathcal{P}$, and let us prove that $f \in \mathcal{F}$. Since every strict subnetwork of f is in $\mathcal{P}$, f has no strict subnetworks in $\mathcal{F}$. So if f is not in $\mathcal{F}$ then $\textsc{sub}(f) \cap \mathcal{F} = \emptyset$ and we deduce that $f \in \mathcal{P}$, a contradiction. Thus $f \in \mathcal{F}$, so $\mathcal{F}$ contains all the networks critical for $\neg\mathcal{P}$. □

Let $\mathcal{P}_{=1}$ be the set of networks f such that each subnetwork of f has a unique fixed point, and let $\mathcal{F}_{=1}$ be the smallest set of forbidden subnetworks for $\mathcal{P}_{=1}$. Let $\mathcal{F}_{\text{ESD}}$ and $\mathcal{F}_{\text{OSD}}$ be the set of *critical* even- and odd-self-dual networks, respectively, and let $\mathcal{F}_{\text{EOSD}} = \mathcal{F}_{\text{ESD}} \cup \mathcal{F}_{\text{OSD}}$.

Remark 8. A lot of even- or odd-self-dual networks are not critical. For instance, the network f on $\{1, 2, 3\}$ defined by $f_1(x) = x_1 \oplus x_2 \oplus x_3$ and $f_2(x) = f_3(x) = x_1$ is even-self-dual, but it contains two even-self-dual strict subnetworks and two odd-self-dual strict subnetworks.

Corollary 5. $\mathcal{F}_{=1} = \mathcal{F}_{\text{EOSD}}$.

Proof. If $f \in \mathcal{F}_{\text{EOSD}}$, then not strict subnetwork of f is in $\mathcal{F}_{\text{EOSD}}$ and according to Theorem 8, each strict subnetwork of f is in $\mathcal{P}_{=1}$. Since $f \notin \mathcal{P}_{=1}$ (because f has zero or two fixed points), f is critical for $\neg\mathcal{P}_{=1}$, and it follows from the previous proposition that $f \in \mathcal{F}_{=1}$. Thus $\mathcal{F}_{\text{EOSD}} \subseteq \mathcal{F}_{=1}$. Now, by Theorem 8, $\mathcal{F}_{\text{EOSD}}$ is a set of forbidden subnetworks for $\mathcal{P}$, and we deduce from the previous proposition that $\mathcal{F}_{=1} \subseteq \mathcal{F}_{\text{EOSD}}$. □

Let $\mathcal{P}_{\leq 1}$ (resp. $\mathcal{P}_{\geq 1}$) be the set of networks f such that each subnetwork of f has at most (resp. at least) one fixed point; and let $\mathcal{F}_{\leq 1}$ and $\mathcal{F}_{\geq 1}$ be the smallest sets of forbidden subnetworks for $\mathcal{P}_{\leq 1}$ and $\mathcal{P}_{\geq 1}$, respectively. In the light of the "proof by dichotomy" of Robert's theorem (given by Theorem 3) it is tempting to try to deduce that $\mathcal{F}_{\text{EOSD}}$ is the smallest set of forbidden subnetworks for $\mathcal{P}_{=1}$ from the forbidden sets $\mathcal{F}_{\leq 1}$ and $\mathcal{F}_{\geq 1}$. But this is not so simple. Indeed, $\mathcal{F}_{\leq 1} \cup \mathcal{F}_{\geq 1}$ is clearly a set of forbidden subnetworks for $\mathcal{P}_{\leq 1} \cap \mathcal{P}_{\geq 1} = \mathcal{P}_{=1}$, thus $\mathcal{F}_{\text{EOSD}} \subseteq \mathcal{F}_{\leq 1} \cup \mathcal{F}_{\geq 1}$, but the inclusion is strict: A lot of networks critical for $\neg\mathcal{P}_{\leq 1}$ or $\neg\mathcal{P}_{\geq 1}$ are not critical for $\neg\mathcal{P}_{=1}$ (because any network that is critical for $\neg\mathcal{P}_{\leq 1}$ (resp. $\neg\mathcal{P}_{\geq 1}$) and that contains a subnetworks with no (resp. multiple) fixed point is not critical for $\neg\mathcal{P}_{=1}$). Examples are given below.

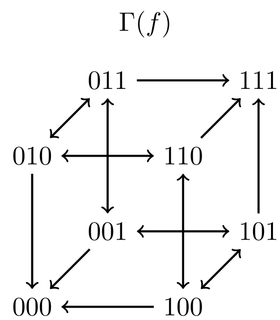
However, in Section 9, we will see that, if we consider the class of non-expansive networks, then $\mathcal{F}_{\text{ESD}} = \mathcal{F}_{\leq 1}$ and $\mathcal{F}_{\text{OSD}} = \mathcal{F}_{\geq 1}$, so that the the equality $\mathcal{F}_{\text{EOSD}} = \mathcal{F}_{\leq 1} \cup \mathcal{F}_{\geq 1}$ holds. Also, in Section 10, we will se that $\mathcal{F}_{\text{ESD}} = \mathcal{F}_{\leq 1}$ for another class of networks (the conjunctive networks), and we will leave the equality $\mathcal{F}_{\text{OSD}} = \mathcal{F}_{\geq 1}$ has an open problem for this class.

Remark 9. f is critical for $\neg\mathcal{P}_{\leq 1}$ if and only if f has at least two fixed points, and every strict subnetwork of f has at most one fixed points. In other words, $\mathcal{F}_{\leq 1}$ *is the set of 2-critical networks.* And similarly, $\mathcal{F}_{\geq 1}$ *is the set of 0-critical networks.*

Among network properties closed for subnetworks, $\mathcal{P}_{=1}$, $\mathcal{P}_{\leq 1}$ and $\mathcal{P}_{\geq 1}$ are not "very strong", and this is why it is interesting to characterize them in terms of forbidden subnetworks. By opposition, closed property as $\mathcal{P}_{>1}$ (every subnetwork has at least two fixed points) or $\mathcal{P}_{<1}$ (every subnetwork has no fixed points) are not interesting. To see this, consider the two one-dimensional constant networks $\text{ZERO}(x) = 0$ and $\text{ONE}(x) = 1$. Clearly ZERO and ONE have a unique fixed point and are thus critical for $\mathcal{P}_{>1}$ or $\mathcal{P}_{<1}$. Consequently, ZERO and ONE are in the smallest forbidden set of subnetworks for $\mathcal{P}_{>1}$ and $\mathcal{P}_{<1}$. But it is easy to see that networks without ZERO or ONE as subnetwork are (exactly) networks f such that $\tilde{f}$ is a constant, and restrict our attention to this type of networks is not interesting. Actually, even if only ZERO or only ONE is forbidden, the resulting networks are too particular to be interesting. In other words: *Interesting closed properties must be satisfied by* ZERO *and* ONE. An interesting property different from $\mathcal{P}_{=1}$, $\mathcal{P}_{\leq 1}$ and $\mathcal{P}_{\geq 1}$, is for example "each subnetwork has an asynchronous state graph which describes a strong convergence toward a unique fixed point". Hence, it would be interesting to characterize the set of forbidden subnetworks for this property.

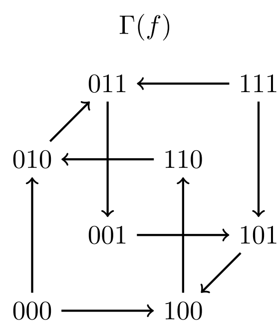
Example 4. The following network f is 2-critical (*i.e.* in $\mathcal{F}_{\leq 1}$) but not even (double arrows indicate cycles of length two):

x	$f(x)$	$\tilde{f}(x)$
000	000	000
001	110	111
010	101	111
011	100	111
100	011	111
101	010	111
110	001	111
111	111	000



The following network f is 0-critical (*i.e.* in $\mathcal{F}_{\geq 1}$) but not odd:

x	$f(x)$	$\tilde{f}(x)$
000	110	110
001	101	100
010	011	001
011	001	010
100	110	010
101	100	001
110	010	100
111	001	110



8. Circular networks and non expansive networks

A **positive-circular** (resp. **negative-circular**) network is a network f such that $G(f)$ is a positive (resp. negative) cycle. Positive- and negative-circular networks have been widely studied (*e.g.* [9, 4]) because that are the "simplest non simple networks" in the sense that they are the most simple networks (from a structural point of view) that do not describe a convergence toward a unique fixed point.

In this section, we show that positive-circular (resp negative-circular) networks are even-self-dual (resp. odd-self-dual), and we prove that the converse holds for non-expansive networks (cf. Theorem 9 below). In this way, even- and odd-self-dual network may be seen as generalization of circular networks.

Suppose that f is a circular network. Let σ be the permutation of V that maps every vertex i to the vertex $\sigma(i)$ preceding i in $G(f)$. For each $x \in \mathbb{B}^V$, let us denote by σx the point of $\mathbb{B}^V$ such that $(\sigma x)_i = x_{\sigma(i)}$ for all $i \in V$. Let $s \in \mathbb{B}^V$ be the such that for all $i \in V$, $s_i = 0$ if the arc from $\sigma(i)$ to i is positive and $s_i = 1$ otherwise. Then, for all $x \in \mathbb{B}^V$, we have

$$f(x) = \sigma x \oplus s.$$

We call σ the **permutation of** f and s the **constant of** f. Since $G(f)$ only depends on f, since the couple (σ, s) only depends on $G(f)$ and since f only depends on this couple, these three objects share the same information. In particular the sign of $G(f)$ is "contained" in s: It is positive if $\|s\|$ is even, and negative if $\|s\|$ is odd.

Theorem 9.

1. *f is positive-circular if and only if f is even-self-dual and non-expansive.*
2. *f is negative-circular if and only if f is odd-self-dual and non-expansive.*

We will use the following lemma several times.

Lemma 3. *Let f be networks on V and $I \subseteq V$. Let f' be the network on V defined by $f'(x) = f(x) \oplus e_I$ for all $x \in \mathbb{B}^V$. We have the following properties.*

1. *If f is non-expansive, then f' is non-expansive.*
2. *If f is self-dual, then f' is self-dual.*
3. *If f is even or odd, then f' is even or odd.*
4. *$|Gf'(x)| = |Gf(x)|$ for all $x \in \mathbb{B}^V$.*

Proof. Suppose that f is non-expansive, and let $x, y \in \mathbb{B}^V$. Then

$$d(f'(x), f'(y)) = d(f(x) \oplus e_I, f(y) \oplus e_I) = d(f(x), f(y)) \leq d(x, y).$$

thus f' is non-expansive.

If f is self-dual then

$$f'(x \oplus 1) = f(x \oplus 1) \oplus e_I = f(x) \oplus 1 \oplus e_I = f'(x) \oplus 1$$

thus f' is self-dual.

Suppose that f is even. For all $x \in \mathbb{B}^V$, we have $\tilde{f}'(x) = f'(x) \oplus x = f(x) \oplus e_I \oplus x = \tilde{f}(x) \oplus e_I$, and since $\tilde{f}(x)$ is even, we deduce that $\tilde{f}'(x)$ and $|I|$ have the same parity. Thus all the points of $\tilde{f}'(\mathbb{B}^V)$ have the parity of $|I|$. Suppose that $|I|$ is even (resp. odd), and let $z \in \mathbb{B}^V$ be an even (resp. odd). Then $z \oplus e_I$ is even, thus there exists $x \in \mathbb{B}^V$ such that $\tilde{f}(x) = z \oplus e_I$, so

$\tilde{f}'(x) = f'(x) \oplus x = \tilde{f}(x) \oplus e_I = z$. Thus every even (resp. odd) point of $\mathbb{B}^V$ is in $\tilde{f}'(\mathbb{B}^V)$. Thus f' is even if $|I|$ is even, and f' is odd otherwise. The proof is similar if f is odd.

For all $i, j \in V$ and $x \in \mathbb{B}^V$,

$$\begin{aligned}|f'_{ij}(x)| = f'_i(x) \oplus f'_i(x \oplus e_j) = f_i(x) \oplus e_I \oplus f_i(x \oplus e_j) \oplus e_I \\ = f_i(x) \oplus f_i(x \oplus e_j) = |f_{ij}(x)|\end{aligned}$$

and the last point follows. □

Proof of Theorem 9. (Direction ⇒) Let f be a circular with permutation σ and constant s. For all $x, y \in \mathbb{B}^V$, we have

$$d(f(x), f(y)) = \|\sigma x \oplus s \oplus \sigma y \oplus s\| = \|\sigma x \oplus \sigma y\| = \|x \oplus y\| = d(x, y).$$

thus f is non expansive. Also,

$$f(x \oplus 1) = \sigma(x \oplus 1) \oplus s = \sigma x \oplus 1 \oplus s = f(x) \oplus 1$$

thus f is self-dual. We now prove that f is even (resp. odd) if $G(f)$ is positive (resp. negative). We have $\tilde{f}(x) = x \oplus \sigma x \oplus s$ so the parity of $\tilde{f}(x)$ is the parity of $\|x\| + \|\sigma x\| + \|s\|$. Since $\|x\| = \|\sigma x\|$, we deduce that the parity of $\tilde{f}(x)$ is the parity of $\|s\|$. So if $G(f)$ is positive (resp. negative) then the image of $\tilde{f}$ only contains even (resp. odd) points. It remains to prove that if $G(f)$ is positive (resp. negative) then each even (resp. odd) point is in the image of $\tilde{f}$. Suppose that $G(f)$ is positive (resp. negative), and let z be an even (resp. odd) point of $\mathbb{B}^V$. Let $n = |V|$ and let $i_1, i_2, \dots i_n$ be the vertices of $G(f)$ given in the order, so that $\sigma(i_1) = i_n$ and $\sigma(i_{k+1}) = i_k$ for $1 \leq k < n$. Let x be the point of $\mathbb{B}^V$ whose components x_{i_k} are recursively defined as follows, with k decreasing from n to 1:

$$x_{i_n} = z_{i_1}, \qquad x_{i_k} = z_{i_{k+1}} \oplus s_{i_{k+1}} \oplus x_{i_{k+1}} \qquad 1 \leq k < n.$$

Let us prove that $\tilde{f}(x) = z$. For every $1 < k \leq n$, we have

$$\begin{aligned}\tilde{f}_{i_k}(x) = f_{i_k}(x) \oplus x_{i_k} = x_{\sigma(i_k)} \oplus s_{i_k} \oplus x_{i_k} \\ = x_{i_{k-1}} \oplus s_{i_k} \oplus x_{i_k} = (z_{i_k} \oplus s_{i_k} \oplus x_{i_k}) \oplus s_{i_k} \oplus x_{i_k} = z_{i_k}.\end{aligned}$$

It remains to prove that $\tilde{f}_{i_1}(x) = z_{i_n}$. By the definition of x, we have

$$\begin{aligned}x_{i_1} &= (z_{i_2} \oplus s_{i_2}) \oplus x_{i_2} \\ &= (z_{i_2} \oplus s_{i_2}) \oplus (z_{i_3} \oplus s_{i_3}) \oplus x_{i_3} \\ &\vdots \\ &= (z_{i_2} \oplus s_{i_2}) \oplus (z_{i_3} \oplus s_{i_3}) \oplus \dots \oplus (z_{i_n} \oplus s_{i_n}) \oplus z_{i_1} \\ &= (z_{i_2} \oplus z_{i_3} \oplus \dots \oplus z_{i_n} \oplus z_{i_1}) \oplus (s_{i_2} \oplus s_{i_3} \oplus \dots \oplus s_{i_n}).\end{aligned}$$

So $\|z\|$ and $\|x_{i_1} \oplus s_{i_2} \oplus s_{i_3} \oplus \cdots \oplus s_{i_n}\|$ have the same parity, and since $\|z\|$ and $\|s\|$ have the same parity, we deduce that $x_{i_1} = s_{i_1}$. Thus

$$\tilde{f}_{i_1}(x) = f_{i_1}(x) \oplus x_{i_1} = x_{i_n} \oplus s_{i_1} \oplus x_{i_1} = z_{i_1} \oplus s_{i_1} \oplus s_{i_1} = z_{i_1}$$

and it follows that $\tilde{f}(x) = z$. So f is even (resp. odd).

(Direction $\Leftarrow$) We first prove the following property:

(1) Suppose that f is odd-self-dual and non-expansive, and suppose that $f(x) = x \oplus e_i$ for some $x \in \mathbb{B}^V$ and $i \in V$. Then the in-degree of i in $Gf(x)$ is at most one.

Let $x^1 = x$, and for all $k \in \mathbb{N}$, let $x^{k+1} = f(x^k)$. Let $n = |V|$, and for all $1 \leq p \leq n$, let us say that a sequence $i_1, i_2, \ldots, i_p$ is *good* if it is a sequence of p distinct vertices in V such that

$$f(x^k) = x^k \oplus e_{i_k} \qquad 1 \leq k \leq p.$$

Let us prove the following property:

($*$) For all $1 \leq p \leq n$, there exists a good sequence $i_1, i_2, \ldots, i_p$.

Since $f(x) = x \oplus e_i$, this is true for $p = 1$. So suppose that $1 < p \leq n$ and that there exists a good sequence $i_1, i_2, \ldots, i_{p-1}$. Since $x^p = f(x^{p-1}) = x^{p-1} \oplus e_{i_{p-1}}$ we have $d(x^p, x^{p-1}) = 1$, and since f is non-expansive, we deduce that

$$d(f(x^p), x^p) = d(f(x^p), f(x^{p-1})) \leq d(x^p, x^{p-1}) = 1.$$

Since f is odd, f has no fixed point, thus $d(f(x^p), x^p) = 1$, *i.e.* there exists an element of V, that we denote by i_p, such that $f(x^p) = x^p \oplus e_{i_p}$. To complete the induction step, it remains to prove that $i_p \neq i_1, i_2 \ldots, i_{p-1}$. Suppose, for a contradiction, that $i_p = i_k$ with $1 \leq k < p$. Then $\tilde{f}(x^p) = \tilde{f}(x^k) = e_{i_k}$. Since f is self dual $\tilde{f}(x^p \oplus 1) = e_{i_k}$. Thus x^p, x^k and $x^p \oplus 1$ are elements of $\tilde{f}^{-1}(e_{i_k})$. Since f is odd-self-dual, $\tilde{f}^{-1}(e_{i_k})$ contains exactly two elements. Thus $x^p = x^k$ or $x^p \oplus 1 = x^k$, and this is not possible since

$$\begin{aligned} x^p &= x^{p-1} \oplus e_{i_{p-1}} \\ &= x^{p-2} \oplus e_{i_{p-2}} \oplus e_{i_{p-1}} \\ &\vdots \\ &= x^k \oplus e_{i_k} \oplus e_{i_{k+1}} \oplus \cdots \oplus e_{i_{p-2}} \oplus e_{i_{p-1}}. \end{aligned}$$

This prove the induction step and ($*$) follows. So let $i_1, i_2, \ldots, i_n$ be a good sequence. Since $x = x^1$ and $f(x) = x \oplus e_i$, we have $i = i_1$. To prove **(1)**, we will prove that if $Gf(x)$ has an arc from i_k to i, then $k = n$. So let $1 \leq k \leq n$, and suppose that $Gf(x)$ contains an arc from i_k to i. Since f is non-expansive,

$$f(x \oplus e_{i_k}) = f(x) \oplus e_i = x \oplus e_i \oplus e_i = x = (x \oplus e_{i_k}) \oplus e_{i_k}$$

Thus $\tilde{f}(x \oplus e_{i_k}) = \tilde{f}(x^k) = e_{i_k}$. Since f is self-dual $\tilde{f}(x \oplus e_{i_k} \oplus 1) = e_{i_k}$. Thus $x \oplus e_{i_k}$, x^k and $x \oplus e_{i_k} \oplus 1$ are elements of $\tilde{f}^{-1}(e_{i_k})$, and as previously, we deduce that $x^k = x \oplus e_{i_k}$ or $x^k = x \oplus e_{i_k} \oplus 1$. Thus $k > 1$, and since

$$x^k = x \oplus e_{i_1} \oplus e_{i_2} \oplus \cdots \oplus e_{i_{k-1}}$$

we have $x^k \neq x \oplus e_{i_k}$. Thus $x^k = x \oplus e_{i_k} \oplus 1$ so $e_{i_k} \oplus 1 = e_{i_1} \oplus e_{i_2} \oplus \cdots \oplus e_{i_{k-1}}$. If $k < n$ then $(e_{i_k} \oplus 1)_{i_n} = 1$ and $(e_{i_1} \oplus e_{i_2} \oplus \cdots \oplus e_{i_{k-1}})_{i_n} = 0$, a contradiction. Thus $k = n$ and **(1)** is proved.

(2) Suppose that f even-self-dual and non-expansive and suppose that $f(x) = x$ for some $x \in \mathbb{B}^V$. Then $Gf(x)$ is a disjoint union of cycles.

Let $i \in V$. If $f(x \oplus e_i) = x$ then $\tilde{f}(x \oplus e_i) = e_i$ and this is not possible since f is even. Since f is non-expansive, we deduce that there exists $j \in V$ such that $f(x \oplus e_i) = f(x) \oplus e_j$. Then j is the unique out-neighbor of i in $Gf(x)$. Thus we have prove the following:

($*$) Each vertex of $Gf(x)$ has exactly one out-neighbor.

Let $i \in V$, and let h be the network on V defined by $f'(y) = f(y) \oplus e_i$ for all $y \in \mathbb{B}^V$. Since $f(x) = x$, we have $f'(x) = x \oplus e_i$, thus according to Lemma 3, f' is odd-self-dual and non-expansive. So according to **(1)**, i has at most one in-neighbor in $Gf'(x)$, and by Lemma 3, i has at most one in-neighbor in $Gf(x)$. Thus each vertex of $Gf(x)$ has at most one in-neighbor, and using ($*$) we deduce that each vertex of $Gf(x)$ has exactly one in-neighbor. Consequently, $Gf(x)$ is a disjoint union of cycles. This proves **(2)**.

(3) Suppose that f is even- or odd-self-dual and non-expansive. Then $Gf(x)$ is a disjoint union of cycles for all $x \in \mathbb{B}^V$.

Let $x \in \mathbb{B}^V$, and let f' be the network on V defined by $f'(y) = f(y) \oplus \tilde{f}(x)$ for all $y \in \mathbb{B}^V$. Then $f'(x) = f(x) \oplus \tilde{f}(x) = x \oplus \tilde{f}(x) \oplus \tilde{f}(x) = x$ and we deduce from Lemma 3 that f' is even-self-dual and non-expansive. Thus, following **(2)**, $Gf'(x)$ is a disjoint union of cycles, and we deduce from Lemma 3 that $Gf(x)$ is a disjoint union of cycles. This proves **(3)**.

(4) Suppose that f is even- or odd-self-dual and non-expansive. Then $Gf(x) = G(f)$ for all $x \in \mathbb{B}^V$.

Let $x \in \mathbb{B}^V$ and $i, k, l \in V$. Suppose that

$$f_{lk}(x) = s \neq 0 \qquad \text{and} \qquad f_{lk}(x \oplus e_i) \neq s.$$

Since $f_{lk}(x) = f_{lk}(x \oplus e_k)$, we have $k \neq i$, and since, by **(3)**, each vertex of $Gf(x)$ has a unique in-neighbor, we have $f_l(x) = f_l(x \oplus e_i)$. Suppose that $x_k = 0$. Then

$$f_{lk}(x \oplus e_i) = f_l(x \oplus e_i \oplus e_k) - f_l(x \oplus e_i) = f_l(x \oplus e_i \oplus e_k) - f_l(x) \neq s$$

and $f_{lk}(x) = f_l(x \oplus e_k) - f_l(x) = s$. Thus $f_l(x \oplus e_i \oplus e_k) \neq f_l(x \oplus e_k)$, that is, $f_{li}(x \oplus e_k) \neq 0$. Thus $Gf(x \oplus e_k)$ contains both an arc from k to l and from i to

l. Since $i \neq k$, l has at least two in-neighbor in $Gf(x \oplus e_k)$, and this contradicts **(3)**. If $x_k = 1$, we obtain a contradiction with similar arguments. Thus:

$$\forall x \in \mathbb{B}^V,\ \forall i,k,l \in V, \qquad f_{lk}(x) \neq 0\ \Rightarrow f_{lk}(x) = f_{lk}(x \oplus e_i)$$

We deduce that $Gf(x)$ is a subgraph of $Gf(x \oplus e_i)$ and that $Gf(x \oplus e_i)$ is a subgraph of $Gf((x \oplus e_i) \oplus e_i) = Gf(x)$. Thus $G(x) = G(x \oplus e_i)$ for all $x \in \mathbb{B}^V$ and $i \in V$, and as an immediate consequence, $Gf(x) = Gf(y)$ for all $x, y \in \mathbb{B}^V$. This proves **(4)**.

(5) If f is even-self-dual and non-expansive, then $G(f)$ is a positive cycle.

Indeed, following **(3)** and **(4)**, $G(f)$ is a disjoint union of cycles and since f if even-self-dual, f has exactly 2 fixed points. Thus $G(f)$ has only positive cycles (otherwise f would have no fixed point, according to Theorem 2). And since if $G(f)$ is a union of $p \geq 1$ disjoint positive cycle then f has 2^p fixed points, we deduce that $G(f)$ is a positive cycle.

(6) If f is odd-self-dual and non-expansive, then $G(f)$ is a negative cycle.

Let $i \in V$ and let f' be the network on V defined by $f'(x) = f(x) \oplus e_i$ for all $x \in \mathbb{B}^V$. By Lemma 3, f' is even-self-dual and non-expansive. Thus according to **(5)**, $G(f')$ is a cycle. From Lemma 3, we deduce that $G(f)$ is a cycle too. Since f is odd, it has no fixed point, and we deduce that $G(f)$ is a negative cycle. □

As an immediate consequence of this theorem and Corollary 2 we obtain the following:

Corollary 6. *If f is non-expansive, then every subnetwork of f has a unique fixed point if and only if f has no circular subnetwork.*

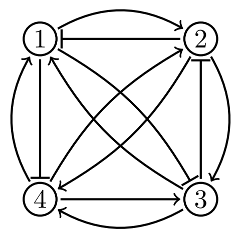
Remark 10. It is easy to check that critical even-self-dual (resp. odd-self-dual) network with at most three components are circular. Below is an example of critical even-self-dual network with four components which is not circular.

Example 5. The following network f on $\{1, 2, 3, 4\}$ is a critical even-self-dual network which is not circular (and which is expansive, since $d(f(0), f(e_i)) \geq 2$ for $i = 1, 2, 3, 4$). Note that the subnetwork f^{40} is the three-dimensional network considered in Examples 1, 2 and 3.

$$\begin{aligned}
f_1(x) &= (\overline{x_2} \wedge x_3 \wedge \overline{x_4}) \vee ((\overline{x_2} \vee x_3) \wedge x_4)\\
f_2(x) &= (\overline{x_3} \wedge x_1 \wedge \overline{x_4}) \vee ((\overline{x_3} \vee x_1) \wedge x_4)\\
f_3(x) &= (\overline{x_1} \wedge x_2 \wedge \overline{x_4}) \vee ((\overline{x_1} \vee x_2) \wedge x_4)\\
f_4(x) &= (x_2 \wedge x_3 \wedge \overline{x_1}) \vee ((x_2 \vee x_3) \wedge x_1)
\end{aligned}$$

x	$f(x)$	$\tilde{f}(x)$
0000	0000	0000
0010	1000	1010
0100	0010	0110
0110	0011	0101
1000	0100	1100
1010	1001	0011
1100	0101	1001
1110	0001	1111
0001	1110	1111
0011	1010	1001
0101	0110	0011
0111	1011	1100
1001	1100	0101
1011	1101	0110
1101	0111	1010
1111	1111	0000

$G(f)$



9. Non-expansive networks

As we have seen in the preceding section, a positive-circular (resp. negative-circular) network f is non-expansive, and it is easy to see that such a network is also 2-critical (resp. 0-critical). The following theorem, the main result of this section, asserts that the converse is true.

Theorem 10.

1. *f is positive-circular if and only if f is 2-critical and non-expansive.*
2. *f is negative-circular if and only if f is 0-critical and non-expansive.*

Even if the two points of this theorem seem similar (symmetrical), their proofs are very different. The proof of the first is rather direct and uses Theorem 8 and a part of Theorem 9 (non-expensive even-self-dual networks are positive-circular). The proof of the second points is independent of previous results. It consists in visiting each point of $\mathbb{B}^V$ in a very special order. In both cases, the following lemma will be useful.

Lemma 4. *Let f be networks on V and $I \subseteq V$. Let f' be the network on V defined by $f'(x) = f(x \oplus e_I) \oplus e_I$ for all $x \in \mathbb{B}^V$. We have the following properties.*

1. *If f is non-expansive, then f' is non-expansive.*
2. *If f is 2-critical, then f' is 2-critical.*
3. *If f is 0-critical, then f' is 0-critical.*
4. $|G(f')| = |G(f)|$.

Proof. Suppose that f is non-expansive, and let $x, y \in \mathbb{B}^V$. Then

$$\begin{aligned} d(f'(x), f'(y)) &= d(f(x \oplus e_I) \oplus e_I, f(y \oplus e_I) \oplus e_I) \\ &= d(f(x \oplus e_I), f(y \oplus e_I)) \leq d(x \oplus e_I, y \oplus e_I) = d(x, y). \end{aligned}$$

thus f' is non-expansive.

Let J be a non-empty subset of V and let h be the subnetwork of f induced by $z \in \mathbb{B}^{V \setminus J}$. Let h' be the network on J defined by $h'(y) = h(y \oplus e_{I \cap J}) \oplus e_{I \cap J}$ for all $y \in \mathbb{B}^J$. Let $x \in \mathbb{B}^V$ be such that $x_{-J} = z \oplus e_{I \setminus J}$. We have

$$h'(x|_J) = g(x|_J \oplus e_{I \cap J}) \oplus e_{I \cap J} = h((x \oplus e_I)|_J) \oplus e_{I \cap J}$$

Since

$$(x \oplus e_I)_{-J} = x_{-J} \oplus e_{I \setminus I} = z \oplus e_{I \setminus J} \oplus e_{I \setminus I} = z$$

we have

$$h((x \oplus e_I)|_J) \oplus e_{I \cap J} = f(x \oplus e_I)|_J \oplus e_{I \cap J} = (f(x \oplus e_I) \oplus e_I)|_J = f'(x)|_J.$$

Thus $h'(x|_J) = f'(x)|_J$ for all $x \in \mathbb{B}^V$ be such that $x_{-J} = z \oplus e_{I \setminus J}$, *i.e.* h' is the subnetwork of f' induced by $z \oplus e_{I \setminus J}$. Since it is clear that h and h' have the same number of fixed points, *2.* and *3.* are proved.

For all $i, j \in V$ and $x \in \mathbb{B}^V$,

$$\begin{aligned} |f'_{ij}(x)| = f'_i(x) \oplus f'_i(x \oplus e_j) &= f_i(x \oplus e_I) \oplus e_I \oplus f_i(x \oplus e_I \oplus e_j) \oplus e_I \\ &= f_i(x \oplus e_I) \oplus f_i(x \oplus e_I \oplus e_j) = |f_{ij}(x \oplus e_I)| \end{aligned}$$

thus $|Gf'(x)| = |Gf(x \oplus e_I)|$ and *4.* follows. □

Proof of Theorem 10. (Direction ⇒ of 1. and 2.) Suppose that f is positive-circular (resp. negative-circular). According to Theorem 9, f is non-expansive, and according to the same theorem, it is even-self-dual (resp. odd-self-dual), thus it has two (resp. no) fixed points. If h is a strict subnetwork of f, then $G(h)$ is a strict subgraph of $G(f)$, thus it is acyclic, and by Robert's theorem, h has a unique fixed point. Thus f is 2-critical (resp. 0-critical).

(Direction ⇐ of 1.) We first need the following property:

(1) Suppose that f is non-expansive. If $f(0) = 0$ and $f(1) = 1$ then $\|x\| = \|f(x)\|$ for all $x \in \mathbb{B}^V$.

Indeed, under these hypothesis,

$$\|f(x)\| = d(0, f(x)) = d(f(0), f(x)) \leq d(0, x) = \|x\|$$

and

$$|V| - \|f(x)\| = d(1, f(x)) = d(f(1), f(x)) \leq d(1, x) = |V| - \|x\|.$$

thus $\|f(x)\| \geq \|x\|$ and it follows that $\|f(x)\| = \|x\|$.

(2) Suppose that f is non-expansive. Suppose also that $f(0) = 0$ and $f(1) = 1$. Let I be a non-empty subset of V. Let $z \in \mathbb{B}^{V\setminus I}$ and let h be the subnetwork of f induced by z. If $h(1) = h(0) \oplus 1$ then $h(0) = 0$.

Let z^0 and z^1 denotes the points of $\mathbb{B}^V$ such that

$$z^0|_I = 0, \qquad z^1|_I = 1, \qquad z^0_{-I} = z^1_{-I} = z.$$

So $h(0) = f(z^0)|_I$ and $h(1) = f(z^1)|_I$. Suppose that $h(1) = h(0) \oplus 1$. Then

$$\begin{aligned} d(f(z^0), f(z^1)) &= d(f(z^0)_{-I}, f(z^1)_{-I}) + d(f(z^0)|_I, f(z^1)|_I) \\ &= d(f(z^0)_{-I}, f(z^1)_{-I}) + d(h(0), h(1)) \\ &= d(f(z^0)_{-I}, f(z^1)_{-I}) + d(h(0), h(0) \oplus 1) \\ &= d(f(z^0)_{-I}, f(z^1)_{-I}) + |I|. \end{aligned}$$

Since f is non-expansive

$$d(f(z^0), f(z^1)) = d(f(z^0)_{-I}, f(z^1)_{-I}) + |I| \leq d(z^0, z^1) = |I|$$

thus $d(f(z^0)_{-I}, f(z^1)_{-I}) = 0$, that is $f(z^0)_{-I} = f(z^1)_{-I} = y$ for some $y \in \mathbb{B}^{V\setminus I}$. Since $f(0) = 0$ and $f(1) = 1$, it follows from **(1)** that

$$\|z\| = \|z^0\| = \|f(z^0)\| = \|f(z^0)|_I\| + \|f(z^0)_{-I}\| = \|h(0)\| + \|y\|$$

and

$$\begin{aligned} |I| + \|z\| = \|z^1\| &= \|f(z^1)\| = \|f(z^1)|_I\| + \|f(z^1)_{-I}\| \\ &= \|h(1)\| + \|y\| = \|h(0) \oplus 1\| + \|y\| = |I| - \|h(0)\| + \|y\|. \end{aligned}$$

Thus

$$2\|z\| = \|z^0\| + \|z^1\| - |I| = 2\|y\|.$$

Hence $\|z\| = \|y\|$ and since $\|z\| = \|h(0)\| + \|y\|$, and it follows that $\|h(0)\| = 0$. This prove **(2)**.

We are now in position to prove that 2-critical non-expansive networks are positive-circular. Suppose that f is 2-critical and non-expansive. Let x be a fixed point of f. Let f' be the network on V defined by $f'(y) = f(y \oplus x) \oplus x$ for all $y \in \mathbb{B}^V$. Then $f'(0) = f(x) \oplus x = x \oplus x = 0$. Furthermore, by Lemma 4, f' is 2-critical (so $f'(1) = 1$) and f' is non-expansive. Suppose that f' has a self-dual strict subnetwork h. Then following **(2)**, we have $h(0) = 0$ and thus $h(1) = 1$, so f' is not 2-critical, a contradiction. We deduce that f' has no self-dual strict subnetwork, and since it has two fixed points, we deduce from Theorem 8 that f' is even-self-dual. Thus, according to Theorem 9, $G(f')$ is a positive cycle. It follows from Lemma 4 that $G(f)$ is a cycle, and since f has two fixed points, $G(f)$ is a positive cycle.

(Direction $\Leftarrow$ of point 2.) We begin with the following fact.

(3) If f is non-expansive and 0-critical, then for all $i \in V$ there exists $x, y \in \mathbb{B}^V$ with $x_i \neq y_i$ such that $\tilde{f}(x) = \tilde{f}(y) = e_i$.

Let $i \in V$ and $\alpha \in \mathbb{B}$. Since f is 0-critical, the immediate subnetwork $f^{i\alpha}$ has at least one fixed point. Thus there exists $x \in \mathbb{B}^V$ with $x_i = \alpha$ such that $f(x)_{-i} = f^{i\alpha}(x_{-i}) = x_{-i}$. Hence, $f(x) = x$ or $f(x) = x \oplus e_i$, and since f has no fixed point, we deduce that $f(x) = x \oplus e_i$. Thus $\tilde{f}(x) = e_i$, and **(3)** follows.

(4) If f is non-expansive and 0-critical then for all $i \in V$ and $x, y \in \mathbb{B}^V$:

$$\tilde{f}(x) = \tilde{f}(y) = e_i \text{ and } x_i \neq y_i \quad \Rightarrow \quad x = y \oplus 1.$$

Suppose that $\tilde{f}(x) = \tilde{f}(y) = e_i$ and $x_i \neq y_i$. Suppose that there exists j such that $x_j = y_j = \alpha$. Then $f^{j\alpha}(x_{-j}) = x_{-j} \oplus e_i$ and $f^{j\alpha}(y_{-j}) = y_{-j} \oplus e_i$. Since f is 0-critical, $f^{j\alpha}$ has a fixed point z. If $x_i = z_i$ then

$$d(f^{j\alpha}(x_{-j}), f^{j\alpha}(z)) = d(x_{-j} \oplus e_i, z) = d(x_{-j}, z) + 1,$$

a contradiction with the fact that $f^{j\alpha}$ is non-expansive. Otherwise $y_i = z_i$ so

$$d(f^{j\alpha}(y_{-j}), f^{j\alpha}(z)) = d(y_{-j} \oplus e_i, z) = d(y_{-j}, z) + 1$$

and we obtain the same contradiction. Consequently, there is no j such that $x_j = y_j$. So $x = y \oplus 1$ and **(4)** is proved.

(5) Suppose that every 0-critical non-expansive network f such that $f(0) = e_i$ for some $i \in V$ is negative circular. Then every 0-critical non-expansive network is negative circular.

Indeed, let f be 0-critical and non-expansive. By **(3)** there exists $i \in V$ and $x \in \mathbb{B}^V$ such that $f(x) = x \oplus e_i$. Let f' be the network on V defined by $f'(y) = f(y \oplus x) \oplus x$ for all $y \in \mathbb{B}^V$. By Lemma 4, f' is 0-critical and non-expansive. Furthermore, $f'(0) = f(x) \oplus x = x \oplus e_i \oplus x = e_i$. Thus, by hypothesis, $G(f')$ is a negative cycle. It follows from Lemma 4 that $G(f)$ is a cycle, and since f has no fixed points, $G(f)$ is a negative cycle. This proves **(5)**.

So according to **(5)**, we can assume, without loss of generality, the following hypothesis:

(H) $f(0) = e_i$ for some $i \in V$.

Also, in the all following, we use the following notations:

$$n = |V|, \; x^1 = 0 \; \text{ and } \; x^{k+1} = f(x^k) \; \text{ for all } k \in \mathbb{N}.$$

We first prove the following property (using arguments similar to the ones introduced in claim **(1)** of the proof of Theorem 9).

(6) For all $k \geq 1$, there exists $i_k \in V$ such that $f(x^k) = x^k \oplus e_{i_k}$, and the resulting sequence $i_1 i_2 i_3 \dots$ is a periodic sequence of period n.

We prove this by induction on k. The case $k = 1$ is given by the the hypothesis **(H)**, so suppose that $k > 1$. Then $x^p = f(x^{p-1}) = x^{p-1} \oplus e_{i_{p-1}}$ thus $d(x^p, x^{p-1}) = 1$, and since f is non-expansive, we deduce that

$$d(f(x^p), x^p) = d(f(x^p), f(x^{p-1})) \leq d(x^p, x^{p-1}) = 1.$$

Since f has no fixed point $d(f(x^p), x^p) = 1$ so there exists $i_k \in V$ such that $f(x^p) = x^p \oplus e_{i_p}$. We now prove that $i_1 i_2 i_3 \dots$ is a periodic sequence of period n. Let $k \geq 1$. Suppose that there exists $l \geq 1$ such that $i_k = i_{k+l}$, and let l be minimal for this property. Then $\tilde{f}(x^k) = \tilde{f}(x^{k+l}) = e_{i_k}$. Since

$$x^{k+l} = x^k \oplus e_{i_k} \oplus e_{i_{k+1}} \oplus \cdots \oplus e_{i_{k+l-1}}$$

and since $i_k \neq i_{k+p}$ for all $1 \leq p < l$ we have $x^{k+l}_{i_k} \neq x^k_{i_k}$ thus following **(4)**, $x^{k+l} = x^k \oplus 1$. Consequently, $l = n$. Thus, the sequence $i_1 i_2 i_3 \dots$ has period n and **(6)** is proved.

(7) As an immediate consequence of **(6)**, we have

$$\begin{array}{lll}
x^1 & = & 0 \\
x^2 & = & e_{i_1} \\
x^3 & = & e_{i_1} \oplus e_{i_2} \\
 & \vdots & \\
x^k & = & e_{i_1} \oplus e_{i_2} \oplus e_{i_3} \oplus \cdots \oplus e_{i_{k-1}} \\
 & \vdots & \\
x^{n+1} & = & e_{i_1} \oplus e_{i_2} \oplus e_{i_3} \oplus \cdots \oplus e_{i_{k-1}} \oplus \cdots \oplus e_{i_n} \quad = \quad 1
\end{array}$$

and

$$\begin{array}{lll}
x^{n+1} & = & 1 \\
x^{n+2} & = & 1 \oplus e_{i_1} \\
x^{n+3} & = & 1 \oplus e_{i_1} \oplus e_{i_2} \\
 & \vdots & \\
x^{n+k} & = & 1 \oplus e_{i_1} \oplus e_{i_2} \oplus e_{i_3} \oplus \cdots \oplus e_{i_{k-1}} \\
 & \vdots & \\
x^{2n+1} & = & 1 \oplus e_{i_1} \oplus e_{i_2} \oplus e_{i_3} \oplus \cdots \oplus e_{i_{k-1}} \oplus \cdots \oplus e_{i_n} \quad = \quad 0.
\end{array}$$

Let h be the negative-circular network on V such that $G(h)$ is the negative cycle with a negative arc from i_n to $i_{n+1} = i_1$ and a positive arc from i_k to i_{k+1} for all $1 \leq k < n$. In this way, for all $x \in \mathbb{B}^V$,

$$h_{i_1}(x) = x_{i_n} \oplus 1, \qquad h_{i_k}(x) = x_{i_{k-1}} \quad 1 < k \leq n.$$

We will prove that $h = f$, using several times the following easy tow next properties.

(8) For all $x \in \mathbb{B}^V$ and $1 \le k < l \le n$,

$$\left.\begin{array}{l} f(x \oplus e_{i_k}) = h(x \oplus e_{i_k}) \\ f(x \oplus e_{i_l}) = h(x \oplus e_{i_l}) \end{array}\right\} \quad \Rightarrow \quad \begin{array}{l} f(x) = h(x) \text{ or} \\ \quad f(x) = h(x) \oplus e_{i_{k+1}} \oplus e_{i_{l+1}}. \end{array}$$

Since f is non expansive,

$$\begin{array}{rcccl} d(f(x), f(x \oplus e_{i_k})) & = & d(f(x), h(x \oplus e_{i_k})) & \le & 1 \\ d(f(x), f(x \oplus e_{i_l})) & = & d(f(x), h(x \oplus e_{i_l})) & \le & 1 \end{array}$$

Also $h(x \oplus e_{i_k}) = h(x) \oplus e_{i_{k+1}}$ and $h(x \oplus e_{i_l}) = h(x) \oplus e_{i_{l+1}}$. From $k \neq l$ it comes that $d(h(x \oplus e_{i_k}), h(x \oplus e_{i_l})) = 2$ and thus

$$\begin{array}{rcl} d(f(x), h(x) \oplus e_{i_{k+1}}) & = & 1 \\ d(f(x), h(x) \oplus e_{i_{k+l}}) & = & 1 \end{array}$$

Hence, there exists p, q such that

$$\begin{array}{rcl} f(x) & = & h(x) \oplus e_{i_{k+1}} \oplus e_{i_p} \\ f(x) & = & h(x) \oplus e_{i_{l+1}} \oplus e_{i_q} \end{array}$$

Thus if $f(x) \neq h(x)$ then $i_p = i_{l+1}$ and $i_q = i_{k+1}$. This proves **(8)**.

(9) For all $x \in \mathbb{B}^V$ and $1 \le k < l < p \le n$,

$$\left.\begin{array}{l} f(x \oplus e_{i_k}) = h(x \oplus e_{i_k}) \\ f(x \oplus e_{i_l}) = h(x \oplus e_{i_l}) \\ f(x \oplus e_{i_p}) = h(x \oplus e_{i_p}) \end{array}\right\} \quad \Rightarrow \quad f(x) = h(x).$$

Indeed, if $f(x) \neq h(x)$, then according to **(8)**,

$$\begin{array}{rcl} f(x) & = & h(x) \oplus e_{i_{k+1}} \oplus e_{i_{l+1}} \\ f(x) & = & h(x) \oplus e_{i_{k+1}} \oplus e_{i_{p+1}} \end{array}$$

thus $i_{l+1} = i_{p+1}$, a contradiction. This proves **(9)**.

(10) If $x \in \mathbb{B}^V$ and $x_{i_1} > x_{i_n}$ then $f(x) = h(x)$.

Let $x \in \mathbb{B}^V$ be such that $x_{i_1} = 1$ and $x_{i_n} = 0$. Consider the sequence $s(x) = x_{i_1} x_{i_2} \dots x_{i_n}$, and decompose this sequence into maximal subsequences with only 1 or only 0, in the following way:

$$s(x) = \underbrace{11\cdots 11}_{s(x)^1}\underbrace{00\cdots 00}_{s(x)^2}\underbrace{11\cdots 11}_{s(x)^3}\underbrace{00\cdots 00}_{s(x)^4}11\cdots // \cdots 11\underbrace{00\cdots 00}_{s(x)^{t(x)}}.$$

Clearly, $t(x)$ is even and $t(x) \ge 2$ (since $x_{i_1} > x_{i_n}$). For each $1 \le p \le t(x)$, let $|s(x)^p|$ denote the length of $s(x)^p$.

(a) Suppose that $t(x) = 2$. Then $s(x)$ has the following form:

$$s(x) = x_{i_1} x_{i_2} \dots x_{i_n} = \underbrace{11 \cdots 11}_{s(x)^1} \underbrace{00 \cdots 00}_{s(x)^2}.$$

Let k be such that x_{i_k} is the first element of $s(x)^2$ (or equivalently, the first zero of $s(x)$). Then $x = e_{i_1} \oplus e_{i_2} \oplus \cdots \oplus e_{i_{k-1}}$, so $h(x) = x \oplus e_{i_k}$. Following **(7)** we have $x = x^k$ and $f(x^k) = x^k \oplus e_{i_k}$ thus $f(x) = h(x)$.

(b) Suppose that $t(x) = 4$. Then $s(x)$ has the following form:

$$s(x) = x_{i_1} x_{i_2} \dots x_{i_n} = \underbrace{11 \cdots 11}_{s(x)^1} \underbrace{00 \cdots 00}_{s(x)^2} \underbrace{11 \cdots 11}_{s(x)^3} \underbrace{00 \cdots 00}_{s(x)^4}.$$

We show that $f(x) = h(x)$ by induction on $|s(x)^2|$ and then on $|s(x)^3|$. Let x_{i_k} be the first element of $s(x)^2$, let x_{i_l} be the first element of $s(x)^3$, and let x_{i_p} be the last element of $s(x)^3$, so that:

$$s(x)^2 = x_{i_k} x_{i_{k+1}} \cdots x_{i_{l-1}} \qquad s(x)^3 = x_{i_l} x_{i_{l+1}} \cdots x_{i_p}.$$

- Suppose that $|s(x)^2| = 1$. Assume first that $|s(x)^3| = 1$. In this situation, $s(x)^2 = x_{i_k}$, $s(x)^3 = x_{i_{k+1}}$ and

  $$x = e_{i_1} \oplus e_{i_2} \oplus \cdots \oplus e_{i_{k-1}} \oplus e_{i_{k+1}}$$

  so that
  $$h(x) = x \oplus e_{i_k} \oplus e_{i_{k+1}} \oplus e_{i_{k+2}}.$$

  Also $t(x \oplus e_{i_k}) = 2$ and $t(x \oplus e_{i_{k+1}}) = 2$, and from **(a)** it follows that $f(x \oplus e_{i_k}) = h(x \oplus e_{i_k})$ and $f(x \oplus e_{i_{k+1}}) = h(x \oplus e_{i_{k+1}})$. Consequently, according to **(8)**, we have $f(x) = h(x)$ or $f(x) = h(x) \oplus e_{i_{k+1}} \oplus e_{i_{k+2}}$. In the second case,

  $$\begin{aligned} f(x) &= h(x) \oplus e_{i_{k+1}} \oplus e_{i_{k+2}} \\ &= x \oplus e_{i_k} \oplus e_{i_{k+1}} \oplus e_{i_{k+2}} \oplus e_{i_{k+1}} \oplus e_{i_{k+2}} \\ &= x \oplus e_{i_k}. \end{aligned}$$

  Thus $\tilde{f}(x) = e_{i_k}$. Following **(7)**, $\tilde{f}(x^{n+k}) = e_{i_k}$ and we deduce from **(4)** that $x^{n+k} = x \oplus 1$, which is a contradiction since by, **(7)**,

  $$x^{n+k} = 1 \oplus e_{i_1} \oplus e_{i_2} \oplus e_{i_3} \oplus \cdots \oplus e_{i_{k-1}} = 1 \oplus x \oplus e_{i_{k+1}}.$$

  Consequently, $f(x) = h(x)$. This proves the base case of **(b1)**. For the induction step, assume that $|s(x)^3| > 1$. Then $s(x)^2 = x_{i_k}$, $s(x)^3 = x_{i_{k+1}} \cdots x_{i_p}$ and

  $$x = e_{i_1} \oplus e_{i_2} \oplus \cdots \oplus e_{i_{k-1}} \oplus e_{i_{k+1}} \oplus e_{i_{k+2}} \oplus \cdots \oplus e_{i_p}$$

  so that
  $$h(x) = x \oplus e_{i_k} \oplus e_{i_{k+1}} \oplus e_{i_{p+1}}.$$

Also $t(x \oplus e_{i_k}) = 2$ and we deduce from **(a)** that $f(x \oplus e_{i_k}) = h(x \oplus e_{i_k})$. In addition, $t(x \oplus e_{i_p}) = 4$ and

$$|s(x \oplus e_{i_p})^2| = 1 < |s(x \oplus e_{i_p})^3| = |s(x)^3| - 1.$$

Thus, by induction hypothesis, $f(x \oplus e_{i_p}) = h(x \oplus e_{i_p})$. Hence, according to **(8)** we have $f(x) = h(x)$ or $f(x) = h(x) \oplus e_{i_{k+1}} \oplus e_{i_{p+1}}$. In the second case,

$$\begin{aligned} f(x) &= h(x) \oplus e_{i_{k+1}} \oplus e_{i_{p+2}} \\ &= x \oplus e_{i_k} \oplus e_{i_{k+1}} \oplus e_{i_{p+1}} \oplus e_{i_{k+1}} \oplus e_{i_{p+1}} \\ &= x \oplus e_{i_k}. \end{aligned}$$

Thus $\tilde{f}(x) = e_{i_k}$. Following **(7)**, $\tilde{f}(x^{n+k}) = e_{i_k}$ and we deduce from **(4)** that $x^{n+k} = x \oplus 1$, which is a contradiction since, by **(7)**,

$$\begin{aligned} x^{n+k} &= 1 \oplus e_{i_1} \oplus e_{i_2} \oplus e_{i_3} \oplus \cdots \oplus e_{i_{k-1}} \\ &= 1 \oplus x \oplus e_{i_{k+1}} \oplus e_{i_{k+2}} \oplus \cdots \oplus e_{i_p}. \end{aligned}$$

Consequently, $f(x) = h(x)$.

- Suppose that $|s(x)^2| > 1$. Then $t(x \oplus e_{i_k}) = t(x \oplus e_{i_{l-1}}) = 4$, and $|s(x \oplus e_{i_k})^2| = |s(x \oplus e_{i_{l-1}})^2| < |s(x)^2|$. Thus, by induction hypothesis,

$$\begin{aligned} f(x \oplus e_{i_k}) &= h(x \oplus e_{i_k}) \\ f(x \oplus e_{i_{l-1}}) &= h(x \oplus e_{i_{l-1}}) \end{aligned}$$

Suppose that $|s(x)^3| = 1$ so that $s(x)^3 = x_{i_l}$. Then $t(x \oplus e_{i_l}) = 2$ and we deduce from **(a)** that $f(x \oplus e_{i_l}) = h(x \oplus e_{i_l})$ and from **(9)** it comes that $f(x) = h(x)$. Now, suppose that $|s(x)^3| > 1$. Then $t(x \oplus e_{i_p}) = 4$, $|s(x \oplus e_{i_p})^2| = |s(x)^2|$ and $|s(x \oplus e_{i_p})^3| = |s(x)^3| - 1$, thus, by induction hypothesis,

$$f(x \oplus e_{i_p}) = h(x \oplus e_{i_p})$$

and according to **(9)**, $f(x) = h(x)$.

(c) Suppose that $t(x) \geq 4$. We prove that $f(x) = h(x)$ by induction on $t(x)$ and then on $|s(x)^2| + |s(x)^4|$. The base case $t(x) = 4$ is given by **(b)**. So assume that $t(x) \geq 6$. We use the following notations:

$$\begin{aligned} s(x)^2 &= x_{i_k} x_{i_{k+1}} \cdots x_{i_q} \\ s(x)^4 &= x_{i_l} x_{i_{l+1}} \cdots x_{i_p} \\ s(x)^{t(x)-1} &= x_{i_r} x_{i_{r+1}} \cdots x_{i_s} \end{aligned}$$

- Suppose that $|s(x)^2| + |s(x)^4| = 2$. Then $t(x \oplus e_{i_k}) = t(x \oplus e_{i_l}) = t(x) - 2$. Thus, by induction hypothesis.

$$\begin{aligned} f(x \oplus e_{i_k}) &= h(x \oplus e_{i_k}) \\ f(x \oplus e_{i_l}) &= h(x \oplus e_{i_l}) \end{aligned}$$

We prove that $f(x) = h(x)$ by induction on $|s(x)^{t(x)-1}|$. If $|s(x)^{t(x)-1}| = 1$ then $t(x \oplus e_{i_r}) = t(x) - 2$ and by induction hypothesis,

$$f(x \oplus e_{i_r}) = h(x \oplus e_{i_r}).$$

Thus according to **(9)**, $f(x) = h(x)$. If $|s(x)^{t(x)-1}| > 1$ then $t(x \oplus e_{i_s}) = t(x)$, $|s(x \oplus e_{i_s})^2| + |s(x \oplus e_{i_s})^4| = 2$ and $|s(x \oplus e_{i_s})^{t(x)-1}| < |s(x)^{t(x)-1}|$, thus, by induction hypothesis,

$$f(x \oplus e_{i_s}) = h(x \oplus e_{i_s})$$

and according to **(9)**, $f(x) = h(x)$.

- Suppose that $|s(x)^2| + |s(x)^4| > 2$. Then either $|s(x)^2| \geq 2$ or $|s(x)^4| \geq 2$. Suppose that $|s(x)^2| \geq 2$, the other case being similar. Then $t(x \oplus e_{i_k}) = t(x \oplus e_{i_q}) = t(x)$ and $|s(x \oplus e_{i_k})^2| = |s(x \oplus e_{i_q})^2| < |s(x)^2|$ and $|s(x \oplus e_{i_k})^4| = |s(x \oplus e_{i_q})^4| = |s(x)^4|$, and so, by induction hypothesis,

$$\begin{aligned} f(x \oplus e_{i_k}) &= h(x \oplus e_{i_k}) \\ f(x \oplus e_{i_q}) &= h(x \oplus e_{i_q}). \end{aligned}$$

If $|s(x)^4| = 1$ then $t(x \oplus e_{i_l}) = t(x) - 2$ thus, by induction hypothesis, $f(x \oplus e_{i_l}) = h(x \oplus e_{i_l})$; otherwise, $t(x \oplus e_{i_l}) = t(x)$ and $|s(x \oplus e_{i_l})^2| = |s(x)^2|$ and $|s(x \oplus e_{i_l})^4| < |s(x)^4|$, and so, by induction hypothesis, we have again

$$f(x \oplus e_{i_l}) = h(x \oplus e_{i_l}).$$

Thus, according to **(9)**, $f(x) = h(x)$. This ends the proof of **(10)**.

With similar arguments, we get:

(11) If $x \in \mathbb{B}^V$ and $x_{i_1} < x_{i_n}$ then $f(x) = h(x)$.

Hence, to complete the proof, it remains to prove that if $x_{i_1} = x_{i_n}$ then $f(x) = h(x)$. Assume that $x_{i_1} = x_{i_n} = 0$. We proceed by induction on $\|x\|$. If $\|x\| = 0$ then $f(x) = h(x)$ according to **(7)**. Otherwise, there exists $1 < k < n$ such that $x_{i_k} = 1$. Since $\|x \oplus e_{i_k}\| = \|x\| - 1$, by induction hypothesis,

$$f(x \oplus e_{i_k}) = h(x \oplus e_{i_k}).$$

Now since $(x \oplus e_{i_1})_{i_1} > (x \oplus e_{i_1})_{i_n}$ and $(x \oplus e_{i_n})_{i_1} < (x \oplus e_{i_n})_{i_n}$, according to **(10)** and **(11)** we have

$$\begin{aligned} f(x \oplus e_{i_1}) &= h(x \oplus e_{i_1}) \\ f(x \oplus e_{i_n}) &= h(x \oplus e_{i_n}) \end{aligned}$$

and we deduce from **(9)** that $f(x) = h(x)$. If $x_{i_1} = x_{i_n} = 1$, we prove with similar arguments that $f(x) = h(x)$. Thus $f = h$. □

As a consequence of this theorem and the fact that a network with multiple fixed points (resp. without fixed point) has a 2-critical (resp. 0-critical) subnetwork, we obtain the following "dichotomization" of Corollary 6.

Corollary 7. *Suppose that f is non-expansive.*

1. *Each subnetwork of f has at most one fixed point if and only if f has no positive-circular subnetwork.*
2. *Each subnetwork of f has at least one fixed point if and only if f has no negative-circular subnetwork.*

It is easy to see that, if the maximal in-degree of the global interaction graph $G(f)$ of a network f is at most one, then $Gf(x) = G(f)$ for all $x \in \mathbb{B}^V$. Thus, in particular, if f is circular then $Gf(x) = G(f)$ for all $x \in \mathbb{B}^V$. Proceeding as in Section 5 with this property instead of Proposition 4, we obtain the following corollary. Note that the second point generalizes Theorem 7.

Corollary 8. *Suppose that f is non-expansive.*

1. *If, for every $1 \leq k \leq |V|$, there exists at most $2^k - 1$ points x such that $Gf(x)$ has a chordless positive cycle of length k, then f has at most one fixed points.*
2. *If, for every $1 \leq k \leq |V|$, there exists at most $2^k - 1$ points x such that $Gf(x)$ has a chordless negative cycle of length k, then f has at least one fixed points.*

10. Conjonctive networks

A network f on V is an **and-net** (or **conjunctive network**) if $G(f)$ is simple and if, for every $i \in V$, f_i is the conjunction of the positive and negative inputs of i in $G(f)$, that is: For all $x \in \mathbb{B}^V$, $f_i(x) = 1$ if and only if $G(f)$ has no positive arc $j \to i$ with $x_j = 0$ and no negative arc $j \to i$ with $x_j = 1$. Note that every subnetwork of an and-net is an and-net. Note also that for the class of and-nets, f and $G(f)$ share the same informations.

In this section, we first prove that every 2-critical and-net is positive circular (but we were not able to prove that every 0-critical and-net is negative circular). Then, we show that, for and-nets, the presence of even-self-dual (resp. odd-self-dual) subnetworks can be checked in a very simple way by looking at the chordless positive (resp. negative) cycles of $G(f)$.

Proposition 7.

1. *f if positive-circular if and only if f is an even-self-dual and-net.*
2. *f if negative-circular if and only if f is an odd-self-dual and-net.*

Proof. Suppose that f is positive-circular (negative-circular). Then, by Theorem 9, f is even-self-dual (resp. odd-self-dual), and since each vertex $i \in V$ has exactly one in-neighbor in $G(f)$, f is an and-net.

Suppose that f is an even- or odd-self-dual and-net. Let $i, j, k \in V$, and assume that j and k are distinct in-neighbor of i in $G(f)$. Let $x \in \mathbb{B}^V$ be such that $f_i(x) = 1$. Then $f_i(y) = 0$ for every y such that $y_j \neq x_j$ or $y_k \neq x_k$. So $f_i(x \oplus e_j) = f_i(x \oplus e_j \oplus 1) = 0$, so f is not self-dual, a contradiction. Since f_i is not a constant, we deduce each vertex of $G(f)$ is of in-degree one. According to Proposition 4, each vertex of $G(f)$ is of out-degree at least one. Since the sum of the in-degrees equals the sum of the out-degrees, we deduce that each vertex of $G(f)$ is of in-degree one and out-degree one [3]. In other words, $G(f)$ is a disjoint union of cycles. Let C be a cycle of $G(f)$ with vertex set I. Then, for all $x \in \mathbb{B}^V$,

$$f_i(x \oplus e_I) = f_i(x) \oplus 1 \qquad \forall i \in I,$$

and since $G(f)$ has no arc from I to $V \setminus I$, we deduce that

$$f_i(x \oplus e_I) = f_i(x) \qquad \forall i \in V \setminus I,$$

So $f(x \oplus e_I) = f(x) \oplus e_I$, and thus:

$$\tilde{f}(x \oplus e_I) = (x \oplus e_I) \oplus (f(x) \oplus e_I) = x \oplus f(x) = \tilde{f}(x).$$

and since f is even- or odd-self-dual, we deduce that $x \oplus e_I = x \oplus 1$, that is $I = V$. So $G(f)$ is a cycle, which is positive if f is even, and negative otherwise. □

Using this proposition and Corollary 2 we obtain the following characterization.

Corollary 9. *If f is an and-net, then each subnetwork of f has a unique fixed point if and only if f has no circular subnetworks.*

We will now show that the "unicity part" of this characterization can be obtained under the absence of positive-circular subnetwork.

Theorem 11. *f is positive-circular if and only if f is a 2-critical and-net.*

Proof. If P is a sequence of signed arcs of $G(f)$, we set $s(P) = 0$ if P has an even number of negative arcs, and $s(P) = 1$ if P has an odd number of negative arcs. We first prove the following two properties (which may be of independent interest).

(1) Suppose that f is an and-net. Suppose also that there exists $x \in \mathbb{B}^V$ such that $f(x) = x$ and $f(x \oplus 1) = x \oplus 1$. Let

$$P = (i_1, s_1, i_2), (i_2, s_2, i_3), \ldots, (i_{l-1}, s_{l-1}, i_l), (i_l, s_l, i_{l+1})$$

be a sequence of arcs of $G(f)$. Then $s(P) = x_{i_1} \oplus x_{i_{l+1}}$.

[3] If each vertex of $G(f)$ is of out-degree one, then f is non-expansive, and we can conclude by applying Theorem 9. However, we give here the few additional arguments that makes the proof independent of Theorem 9.

We proceed by induction of the length l of the sequence.

1. Suppose that $l = 1$, that is $P = (i_1, s_1, i_2)$. If $s_1 = 1$, then the arc from i_1 to i_2 is positive and: If $x_{i_1} = 0$ then $f_{i_2}(x) = 0 = x_{i_2}$; and if $x_{i_1} = 1$, then $f_{i_2}(x \oplus 1) = 0 = x_{i_2} \oplus 1$ thus $x_{i_2} = 1$. Hence, in both cases, $x_{i_1} \oplus x_{i_2} = 0 = s(P)$. If $s_1 = -1$, then the arc from i_1 to i_2 is negative so: If $x_{i_1} = 1$ then $f_{i_2}(x) = 0 = x_{i_2}$; and if $x_{i_1} = 0$, then $f_{i_2}(x \oplus 1) = 0 = x_{i_2} \oplus 1$ thus $x_{i_2} = 1$. Hence, in both cases, $x_{i_1} \oplus x_{i_2} = 1 = s(P)$. This prove the base case.
2. Suppose that $l > 1$. Then P can be expressed as the concatenation $P = QQ'$ of two subsequences Q and Q', both of length at most $l - 1$. If q is the length of Q, then, by induction hypothesis, $s(Q) = x_{i_1} \oplus x_{i_{q+1}}$ and $s(Q') = x_{i_{q+1}} \oplus x_{i_{l+1}}$ thus $s(P) = s(Q) \oplus s(Q') = x_{i_1} \oplus x_{i_{q+1}} \oplus x_{i_{q+1}} \oplus x_{i_{l+1}} = x_{i_1} \oplus x_{i_{l+1}}$. This proves **(1)**.

(2) Suppose that f is an and-net. If there exists $x \in \mathbb{B}^V$ such that $f(x) = x$ and $f(x \oplus 1) = x \oplus 1$ then $G(f)$ has no negative cycle.

If C is a cycle of $G(f)$ go length l, and if $P = (i_1, s_1, i_2), (i_2, s_2, i_3), \dots, (i_l, s_l, i_1)$ are the arcs of C given in the order, then following **(1)**, $s(P) = x_{i_1} \oplus x_{i_1} = 0$, thus C has an even number of negative arcs, *i.e.* C is positive. This proves **(2)**.

We are now in position to prove the theorem. By Theorem 10, every positive-circular network is 2-critical, and it is obvious that positive-circular networks are and-nets. So assume that f is a 2-critical and-net. By theorem 8 and Proposition 7, f has a positive- or negative-circular subnetwork h. Following **(2)**, h cannot be negative-circular. Thus h is positive-circular. Thus h has two fixed points, and since f is 2-critical, $h = f$. □

As a consequence of this theorem and the fact that a network with multiple fixed points has a 2-critical subnetwork, we obtain the following characterization.

Corollary 10. *If f is an and-net, then each subnetwork of f has at most one fixed point if and only if f has no positive-circular subnetworks.*

Using again the fact that if f is circular then $Gf(x) = G(f)$ for all $x \in \mathbb{B}^V$, we obtain:

Corollary 11. *Suppose that f is an and-net. If, for every $1 \leq k \leq |V|$ there exists at most $2^k - 1$ points x such that $Gf(x)$ has a chordless positive cycle of length k, then f has at most one fixed points.*

Remark 11. *In view of Theorems 10 and 11, it is tempting to think that* every 0-critical and-net is negative-circular. *But this is false, as showed below. For all $n \geq 4$, let G_n be the digraph with vertex set $V = \{0, 1, \dots, n-1\}$ and such that for all $u \in V$ and $k \in \{1, \pm 2, \pm 3, \dots, \pm\lfloor \frac{1}{2}n \rfloor\}$ there is an arc from u to $u + k \pmod n$. In [5], it is proved that G_n is* kernel-critical*: G_n has no kernel and every strict induced subdigraph has a kernel. Using the correspondence between kernels in digraphs and fixed points in and-nets established in [15], we easily deduce that:* For all $n \geq 4$, the and-net f such that $|G(f)| = G_n$ and such that $G(f)$ has only negative arc is a non-circular 0-critical and-net.

Now, we show how to check if an and-net has or not a circular subnetworks by looking at the chordless cycles of $G(f)$. For that, additional definitions are needed. Let G be a simple interaction graph with vertex set V, and let C be a cycle in it. A vertex $v \in V$ is a **delocalizing vertex** of C if G has both a positive and a negative arcs from v to distinct vertices of C (v can be a vertex of C; in such a case the cycle has two chords of opposite sign starting from v).

Proposition 8 (Richard and Ruet [15])**.** *Suppose that f is an and-net. There exists $x \in \mathbb{B}^V$ such that $Gf(x)$ has a cycle C if and only C is a cycle of $G(f)$ that has no delocalizing vertex in $G(f)$.*

Proposition 9 (Remy and Ruet [10])**.** *Let f be a network on V. Let $x \in \mathbb{B}^V$, and suppose that $Gf(x)$ has a cycle C with vertex set I. If C has no chord in $G(f)$, then the subnetwork of f induced by x_{-I} is a circular network with interaction graph C.*

Proposition 10. *Suppose that f is an and-net. Then f has a circular subnetwork with interaction graph C if and only if C is a cycle of $G(f)$ that has no chord and no delocalizing vertex in $G(f)$.*

Proof. If C a cycle of $G(f)$ without chord and delocalizing vertex in $G(f)$, then the fact that f has a subnetwork with interaction graph C follows from Proposition 8 and Proposition 9.

Suppose that h is a circular subnetwork of f with interaction graph C. Let I be the vertex set of C, $x \in \mathbb{B}^V$, and suppose that h is induced by x_{-I}. Since $Gh(x_{-I}) = G(h) = C$ is a subgraph of $Gf(x)$, we deduce from Proposition 8 that C has no delocalizing vertex in $G(f)$. Suppose, for a contradiction, that C has a chord in $G(f)$, say from j to i. Let $k \neq j$ be the vertex preceding i in C. Let $y \in \mathbb{B}^V$ be such that $y_{-I} = x_{-I}$ and $y_j = 0$ if and only if the chord $j \to i$ is positive. Then $h_i(y_{-I}) = f_i(y) = 0$ and $h_i(y_{-I} \oplus e_k) = f_i(y \oplus e_k) = 0$, thus $Gh(y_{-I})$ has no arc from k to i, a contradiction with the fact that h is circular. □

We are now in position to express conditions in Corollaries 9 and 10 in terms of chordless cycles and delocalizing vertices.

Corollary 12. *Let f be an and-net.*

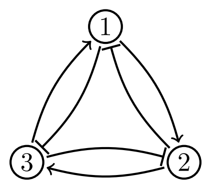
1. *Each subnetwork of f has a unique fixed point if and only if f every chordless cycle of $G(f)$ has a delocalizing vertex.*
2. *Each subnetwork of f has at most one fixed point if and only if f every chordless positive cycle of $G(f)$ has a delocalizing vertex.*

Remark 12. If $G(f)$ has n vertices and c cycles, then the enumeration of these m cycles can be done with time complexity $\mathcal{O}(n^2c)$; see [6] for instance. Since for each cycle the absence chord and delocalizing vertex can be verified in $\mathcal{O}(n^2)$, conditions of Corollary 12 can be verified in time $\mathcal{O}(n^2c)$.

Example 6. [Continuation of Example 1] Take again the network f on $\{1,2,3\}$ defined by:

$$f_1(x) = \overline{x_2} \wedge x_3$$
$$f_2(x) = \overline{x_3} \wedge x_1$$
$$f_3(x) = \overline{x_1} \wedge x_2.$$

This network is an and-net and its global interaction graph $G(f)$ is



It is easy to see that every chordless cycle (*i.e.* cycle of length 2) has a delocalizing vertex. Thus f has no circular subnetwork (cf. Proposition 10). Thus it has no even- or odd-self-dual subnetwork (cf. Proposition 7). Thus each subnetwork of f has a unique fixed point (cf. Corollary 2); see indeed Example 1. Note that the two cycles of length three have no delocalizing vertex, thus these cycles are in $Gf(x)$ for some x; see indeed Example 2.

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Appendix A. Proofs of Theorems 1, 3 and 5

Proof of Theorem 1. We proceed by induction on $|V|$. If $|V| = 1$ the theorem is obvious. So assume that $|V| > 1$, and suppose that $G(f)$ has no cycle. Then $G(f)$ has at least one vertex, say i, without in-neighbor. Hence, $f_i = \text{cst} = \alpha \in \mathbb{B}$. Since $G(f^{i\alpha})$ is a subgraph of $G(f)$, $G(f^{i\alpha})$ has no cycle and by induction hypothesis $f^{i\alpha}$ has a unique fixed point *i.e.* there exists a unique $x \in \mathbb{B}^V$ with $x_i = \alpha$ such that $f^{i\alpha}(x_{-i}) = x_{-i}$. Since $f(x)_{-i} = f^{i\alpha}(x_{-i}) = x_{-i}$ and $f_i(x) = \alpha = x_i$, we deduce that $f(x) = x$. Suppose that f has a fixed point $y \neq x$. Then $f_i(y) = \alpha = y_i$ so $y_{-i} \neq x_{-i}$ and $f^{i\alpha}(y_{-i}) = f(y)_{-i} = y_{-i}$. Thus $f^{i\alpha}$ has a fixed point distinct from x_{-i}, a contradiction. Thus x is the unique fixed point of f. □

Proof of Theorem 3. We proceed by induction on the number of strongly connected components. If $|V| = 1$ then the theorem is obvious. So assume that $|V| > 1$. If $G(f)$ is strongly connected, then the theorem is given by Theorem 2. So suppose that $G(f)$ is not strongly connected, an let $I \subseteq V$ be an initial strongly connected component of $G(f)$ (there is no arc from $V \setminus I$ to I). Let h be the subnetwork of f induced by $0 \in \mathbb{B}^{V \setminus U}$. Let us prove that

$$\forall x \in \mathbb{B}^V, \qquad h(x|_I) = f(x)|_I. \qquad (*)$$

Suppose, for a contradiction, that $h(x|_I) \neq f(x)|_I$ for some $x \in \mathbb{B}^V$, and assume that $\|x\|$ is minimal for this property. Then, since h is induced by the point $0 \in \mathbb{B}^{V\setminus I}$, there exists $j \in V \setminus I$ with $x_j = 1$. Thus $\|x \oplus e_j\| < \|x\|$ so $h(x|_I) = h((x \oplus e_j)|_I) = f(x \oplus e_j)|_I \neq f(x)|_I$. Thus there exists $i \in I$ such that $f_i(x \oplus e_j) \neq f_i(x)$. Thus $G(f)$ has an arc from j to i, a contradiction. This prove $(*)$. We are now in position to complet the induction step.

1. Suppose that $G(f)$ has no positive cycle, and suppose, for a contradiction, that x and y are fixed points of f. Then following $(*)$, $x|_I$ and $y|_I$ are fixed points of h. Since $G(h)$ has no positive cycle, by induction hypothesis, h has at most one fixed point, thus $x|_I = y|_I = z$. Let h' be the subnetwork of f induced by z. By definition, $h'(x_{-I}) = f(x)_{-I}$ and $h'(y_{-I}) = f(y)_{-I}$. Thus x_{-I} and y_{-I} are fixed points of h'. Since $G(h')$ has no positive cycle, by induction hypothesis, h' has at most one fixed point, thus $x_{-I} = y_{-I}$. Thus $x = y$ so f has at most one fixed point.
2. Suppose that $G(f)$ has no negative cycle. Then $G(h)$ has no negative cycle, and by induction hypothesis, h has at least one fixed point $z \in \mathbb{B}^I$. Let h' be the subnetwork of f induced by z. Again, by induction hypothesis, h' has at least one fixed point. Thus, there exists $x \in \mathbb{B}^V$ with $x|_I = z$ such that $x_{-I} = h'(x_{-I}) = f(x)_{-I}$, and by $(*)$ we have $x|_I = z = h(z) = h(x|_I) = f(x)|_I$. Thus x is a fixed point of f.

□

Proof of Theorem 5. The "trick" consists in proving, by induction on $|V|$, the following more general statement:

$(*)$ If $Gf(x)$ has no cycle for all $x \in \mathbb{B}^V$, then the conjugate of f is a bijection (and so f has a unique fixed point).

The case $|V| = 1$ is obvious. So suppose that $|V| > 1$, and suppose that $Gf(x)$ has no cycle for all $x \in \mathbb{B}^V$. Let $i \in V$ and $\alpha \in \mathbb{B}$. For all $x \in \mathbb{B}^V$, $Gf^{i\alpha}(x_{-i})$ is a subgraph of $Gf(x)$, and thus $Gf^{i\alpha}(x_{-i})$ has no cycle. Using the induction hypothesis, we deduce that: *For all $i \in V$ and $\alpha \in \mathbb{B}$, the conjugate of $f^{i\alpha}$ is a bijection.* Now, suppose that $\tilde{f}$ is not a bijection. Then, there exists two distinct points x and y in $\mathbb{B}^V$ such that $\tilde{f}(x) = \tilde{f}(y)$. Let us proved that $x = y \oplus 1$. Indeed, if $x_i = y_i = \alpha$ for some $i \in V$, then $\tilde{f^{i\alpha}}(x_{-i}) = \tilde{f}(x)_{-i} = \tilde{f}(y)_{-i} = \tilde{f^{i\alpha}}(y_{-i})$. Thus the conjugate of $f^{i\alpha}$ is not a bijection, a contradiction. So $x = y \oplus 1$. Since $Gf(x)$ has no cycle, it contains at least one vertex of out-degree 0. In other words, there exists $i \in V$ such that $f(x^{i1}) = f(x^{i0})$. Thus $\tilde{f}(x^{i1})_{-i} = \tilde{f}(x^{i0})_{-i} = \tilde{f}(x)_{-i}$. Hence, setting $\alpha = y_i$, we obtain

$$\tilde{f^{i\alpha}}(x_{-i}) = \tilde{f}(x^{i\alpha})_{-i} = \tilde{f}(x)_{-i} = \tilde{f}(y)_{-i} = \tilde{f}(y^{i\alpha})_{-i} = \tilde{f^{i\alpha}}(y_{-i}).$$

So the conjugate of $f^{i\alpha}$ is not a bijection, a contradiction. Thus $\tilde{f}$ is a bijection and $(*)$ is proved. □

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