On The Usefulness of Carriers, Semi-initiality and Semi-adjuncts for "Institution Independent" Issues

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Abstract:
When defining new specification formalisms, some notions are systematically defined in order to prove some properties of the formalism. Then it seems useful to work at a more abstract level, for several formalisms in a unique proof or definition. That is the goal of specification formalisms.

Terms are among those notions used by many specification formalisms. Together with congruences, they can lead to initial models or left-adjunct to the forgetful functor provided they exist. The goal of this work is to give a semantical characterization of terms, and to give some example of their use in a meta-formalism framework.

Introduction

In the field of algebraic specifications, a lot of different syntax/semantics have been developed. Each of those theories answers (or tries to answer) to some specific aspect related to the activity of formally specifying (observability, subsorting, exception handling, theorem proving issues, modularity issues, etc.). Most of the time, beside the original idea underlying a new theory, the authors have to develop a lot of inevitable formal results which generalize (to the new framework) some "well known classical results". We all know that these "formalities" (as Peter Mosses call them in [Mos89]) are not the most exiting part of our research activity...

They often follow a standard process such as: define signatures Σ, define the category of Σ-models, identify the model of Σ-terms with variables, define formulas (via predicates on Σ-terms, connectives and quantifiers), define the satisfaction relation, define a notion of quotient model (e.g. via congruences), prove the existence of colimits (e.g. via a smallest congruence generated by something), propose a calculus, establish the completeness of the calculus with respect to a particular kind of formulas (e.g. Horn clauses, via a Birkhoff's like proof), and so on and so on.

Since the work of J. Goguen and R. Burstall *introducing institutions* [GB84], it has been established that such standard processes can be formalized themselves (what we call a "meta-formalism"[1]). More recently, other meta-formalisms have been proposed, such as pre-institutions [SS91] or specification frames [EJO93]. A meta-formalism dedicated to algebraic specifications could provide us with a *toolbox* in order to facilitate quick developments of specialized specification frameworks: in software engineering, programming languages are tuned according to the target application, we could reach the same flexibility for algebraic specifications.

In the framework of institutions, A. Tarlecki and D. Sannella proposed several tools for such a toolbox [Tar85][ST88]. In this article our purpose is quite modest: we only aim at modelizing the (meta-)notions of *terms* with variables, *substitutions, smallest congruences*,..., without syntactical restrictions about what a signature is. We believe that these notions are basic and simple enough

[1]similarly to the terminology "méta-mathématiques" proposed by the French Bourbaki group

to get a valuable idea of the "pragmatic" adequacy of our (meta-)definitions (with respect to the practice of an author of algebraic specification theories).

A major difference between our approach and the existing ones is that we start from a refined definition of pre-institution which considers *carriers* of models. Thus, our approach is less abstract than the usual works about institutions; however, the notion of carrier is rather intuitive and we will see that it considerably facilitates the understanding of what a variable is. Moreover, we give up the idea that the model of terms is an initial model because it does not directly cope with some existing theories of algebraic specification.

In our framework, we systematically choose our definitions in such a way that a "naive" author of a specification theory can instantiate the meta-formalism with his (her) own definitions without effort. Else, if the effort to turn his (her) theory into an institution (or whatever) and the effort to instantiate the suitable meta-notions and meta-results are greater than the effort to directly establish the desired results, then (s)he will never use our toolbox, and we would miss the point. Such a method may give rise to less elegant definitions than institutions, pre-institutions or specification frames; we believe it is the price to pay for a pragmatic, usable "toolbox" for an author of a new specification theory.

In Section 1 we discuss two deliberate choices of this article: pre-institutions with carriers and set of terms as models. Ground terms and terms with variables are (meta-)defined in Section 2 and their first applications (evaluations, substitutions) are developed in Section 3. In Section 4, we present a typical meta-result (with respect to what we intend to offer): it is sufficient to establish the existence of a smallest congruence (generated by a relation on carriers) in order to ensure (for free) that initial model and left adjoint to the forgetful functor exist.

1 General Setting

Since there are several meta-formalisms, we have to choose one that fits our needs. Our main purpose is to define a notion of terms, and terms rarely form a model of a specification. They rather form a model of a signature. Thus we need a meta-formalism dealing explicitly with signatures. This leads us to use either institutions [GB84] or pre-institutions [SS91]. In fact the difference between those formalisms deals with the satisfaction of axioms. Since we do not use this concept, we can equally use each of those meta-formalisms.

1.1 Category of *Carriers*

We want to use a notion of variables. In most cases, the so called "sets of variables" are not sets; they are more structured. For instance, in the standard multi-sorted algebraic formalisms, variables are typed, and the *"sets" of variables* are indeed S-sets. We can also notice that models are defined by enriching those structures: the *carriers* of the models are just S-sets.

This leads to the following definition, which is a slight modification of the definition of [SS91]:

Definition 1.1 *A* pre-institutions with carriers *is a pre-institution (i.e. a tuple $\langle Sig, Sen, Mod, \models\rangle$) with a functor $Carr : Sig^{op} \to Cat$, which associates to each signature the category of carriers over the signature, and a natural transformation $|\_| : Mod \to Carr$.*

From now on, we will work in such pre-institutions with carriers.

For each signature Σ there is consequently a new kind of "forgetful functor" $|\_|_\Sigma : \mathrm{Mod}(\Sigma) \to Carr(\Sigma)$. By notation abuse, we will note $|\_|$ instead of $|\_|_\Sigma$ when there is no ambiguity about the signature Σ under consideration.

1.2 Terms form a model

Usually terms are defined in a syntactical way. But in our case, since we do not know anything about the signatures (except the fact they form a category), we cannot define them that way. So we have to define them semantically. In this article, we define a notion of *model of terms* rather than terms themselves.

It is a bit restrictive, since in some theories terms do not form a model of the signature: in [Deo], there are terms of the form "`iter f on X`" which are not individual values but only denote another term where "`iter`" does not appear.

Having a model of terms also helps to define evaluations, since they are carried by morphisms (except for some specification formalisms where morphisms are partial while evaluations are total). Moreover a lot of applications (such as the Birkhoff's completeness proof for the equational calculus) use the fact that terms form a model.

We consider in our work that morphisms are (sort of) functions which preserve the structure of the objects. For instance morphisms of S-sets are compatible with the sorts: they preserve the structure of the S-sets; similarly, being compatible with sorts and operations, morphisms of algebras preserve the structure of algebras. Under such an intuition, it seems reasonable to admit that evaluations of terms are morphisms.

2 Main Definitions

2.1 Ground Terms

To define what we will call "ground terms" in our meta-formalism, we are guided by the properties we want them to have.

- It seems obvious that we can evaluate a ground term in any model of the signature, so the first idea could have been to define the model of ground terms on a signature Σ as the initial object of $\mathrm{Mod}(\Sigma)$ (since we decided to consider models of terms and morphisms of evaluation, cf Section 1.2). But this definition would not be accurate. We may need several possible evaluations from the model of terms (T_Σ) to some of the models. Especially in non-deterministic cases (for instance it is the case of [WM93]), we may need several morphisms $T_\Sigma \to M$ to reflect this non-determinism.

- Moreover, we will say that "Syntax is deterministic": a ground term cannot be evaluated on another, distinct, ground term and two ground terms are equal if and only if they "have the same syntax". In the model described in [GTW78], we say that "$a + b$" and "$b + a$" are two different terms; a different behavior would mean we consider implicit congruences, which could perturb our intuition of terms.

To deal with these requirements, we define a somewhat weaker notion of initiality: the *semi-initiality.*

Definition 2.1 *Let $\mathcal{C}$ be a category. An object $\mathcal{I}$ of $\mathcal{C}$ is said to be* **semi-initial** *in $\mathcal{C}$ if and only if*

1. *For each object M in $\mathcal{C}$, $Hom(\mathcal{I}, M) \neq \emptyset$*

2. *$Hom(\mathcal{I},\mathcal{I}) = \{Id_{\mathcal{I}}\}$*

Lemma 2.2 *All semi-initial objects in $\mathcal{C}$ are isomorphic*

Proof : Let I and I' be two semi-initial objects in $\mathcal{C}$. Since I is semi-initial, there is a morphism $\mu : I \to I'$. Similarly there is a morphism $\nu : I' \to I$. $\mu \circ \nu$ is a morphism $I' \to I'$, and $Hom(I', I') = \{Id_{I'}\}$, so $\mu \circ \nu = Id_{I'}$. Similarly $\nu \circ \mu = Id_I$, so I and I' are isomorphic. □

We will now speak of *the* semi-initial object, since it is unique up to isomorphism.

Moreover, it is straightforward that if $\mathcal{C}$ has an initial object, then this object is also semi-initial in $\mathcal{C}$. Thus:

Corollary 2.3 *If $\mathcal{C}$ has an initial object, then it is the only semi-initial object (up to isomorphism).*

We now can define what we call the model of ground terms.

Definition 2.4 *Given a signature Σ, if $Mod(\Sigma)$ has a semi-initial object, then it is called the model of ground terms, denoted T_Σ .*

Example 2.1 *In [WM93], the* word multi-structure *$MW(\Sigma)$ is not initial in $Mod(\Sigma)$, but it is semi-initial. In this framework, the signatures are made of functions with their arity and $MW(\Sigma)$ is syntactically built as the usual ground terms (trees). However, the function are non-deterministic, and $f_A(a_1, \ldots, a_n)$ denotes the set of all possible values of f in A given the arguments $a_1, \ldots, a_n$ (values of A). Then there is a morphism $\mu : A \to B$ if, for each values $a_1, a_2, \ldots, a_n$ in A, $\mu(f_A(a_1, \ldots, a_n)) \subseteq f_B(\mu(a_1), \ldots, \mu(a_n))$. Consequently there are several morphisms from $MW(\Sigma)$ to the non-deterministic models A; they correspond to all possible choices of the value of $f(t_1, \ldots, t_n)$ in the sets of the form $f_A(\mu(t_1), \ldots, \mu(t_n))$.*

Example 2.2 *In the ADJ case of heterogeneous algebras*[2] *(as in [GTW78]), $Mod(\Sigma)$ has an initial model of ground terms, which is semi-initial (Corollary 2.3), and thus is the model of ground terms according to our definition.*

Counter-example 2.3 *If we do not demand semi-initiality, but only the existence of at least one morphism from T_Σ to any model, then we loose the unicity of this model of ground terms, and obtain more models than we want: in the ADJ example, for any set of variable X, $T_\Sigma(X)$ would be considered as a model of ground terms.*

By convention, we will now call *non deterministic* a formalism which has ground terms and in which there are signatures Σ such that T_Σ is not initial in Mod(Σ).

2.2 Terms with Variables

Before (meta-)defining terms with variables, let us write down the properties we want them to have. We demand them to form a model, for the same reason as ground terms. We want the terms with variables to be an "obvious" extension of ground terms, with the same claim:

- Evaluation of terms is not necessarily deterministic
- "Syntax is deterministic", i.e. each variable occurrence in a term is a place in this term, and the evaluation of a term with variables on another term is entirely characterized by an assignment of its variables.

This gives some conditions on the functor which associates $\mathcal{T}_\Sigma(V)$ to V.

Remark 2.5 *Since we have a sort of forgetful functor $|\_| : Mod(\Sigma) \to Carr(\Sigma)$, it seems interesting to consider a left adjunct to this functor for defining the terms with variables.*

Let us first remark that left adjuncts do not preserve semi-initiality. Let us consider the categories A and B on Fig. 1:

[2] We will simply call this case "ADJ"

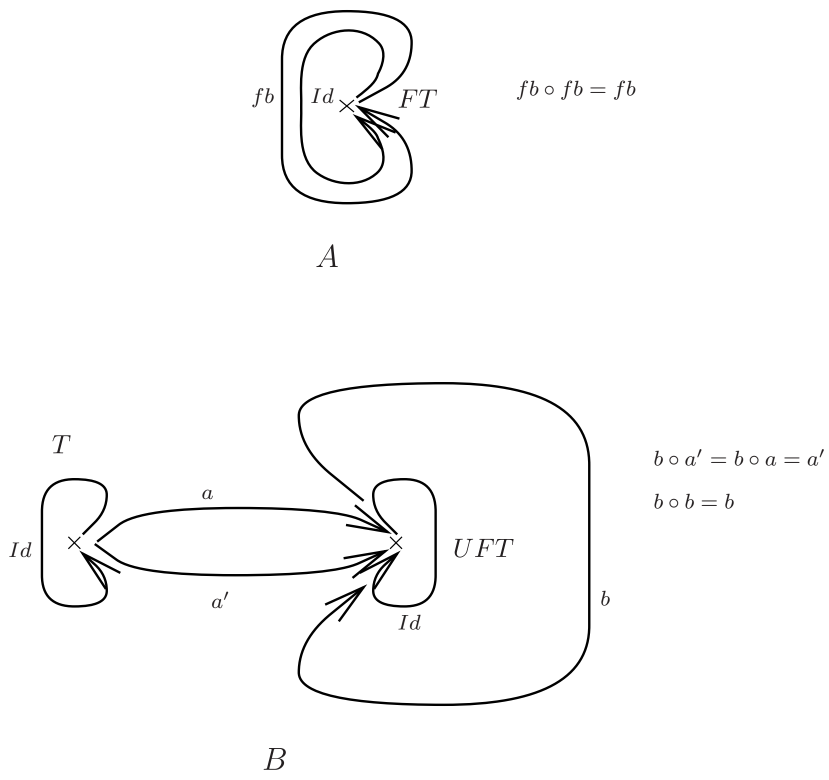

Figure 1: Counter-Example

Let us define the following functors:

$$\begin{aligned} U(FT) &= UFT \\ U(fb) &= b \end{aligned}$$

$$\begin{aligned} F(a) &= Id \\ F(a') &= fb \\ F(b) &= fb \end{aligned}$$

F is a left adjunct to U : We have a natural bijection if there are 2 bijections g et d, ($g : Hom(T, UfT) \to Hom(F(T), F(T))$, $d = g^{-1}$) verifying

1. $\forall \mu : T \to UfT$
   $\forall \alpha T' \to T$
   $g(\mu \circ \alpha) = g(\mu) \circ F(\alpha)$

2. *and* $\nu : F(T) \to B$
   $\beta : B \to B'$
   $d(\beta \circ \nu) = U(\beta) \circ d(\nu)$

Verification of 1: μ is either a or a', and α is Id. Then $g(\mu \circ \alpha) = g(\mu) = g(\mu) \circ F(\alpha)$

Verification of 2: $g(a) = Id$ and $g(a') = b$, so $d(b) = a'$ and $d(Id) = a$. ν and β are either Id or b. We get :

1. *For* $d(\beta \circ \nu)$ *:*

β \ ν	Id	b
Id	a	a'
b	a'	a'

2. *For* $U(\beta) \circ d(\nu)$ *:*

β \ ν	Id	b
Id	a	a'
b	a'	a'

Then F *is left adjunct to* U. T *is obviously semi-initial in* B, *but* $F(T)$ *is not semi-initial in* A.

Fortunately, what we really want is a functor $\mathcal{T}_\Sigma : Carr(\Sigma) \to \mathrm{Mod}(\Sigma)$, such that if $Carr(\Sigma)$ has an initial model O, then $\mathcal{T}_\Sigma(O)$ is semi-initial in $\mathrm{Mod}(\Sigma)$. Then a left adjunct functor would be too strong: in non-deterministic cases, we have no left adjunct functor, but we want to be able to define terms with variables.

We need to define a somewhat weaker notion than adjunction, which we call *semi-adjunction.*

Definition 2.6 *Let* $\mathcal{U} : C_1 \to C_2$ *a functor.* $F : C_2 \to C_1$ *is said to be* left semi-adjunct to $\mathcal{U}$ *if and only if:*

1. *For each* A *object of* C_1 *and each* B *object of* C_2*, there is a natural surjection*

$$s : Hom(F(B), A) \twoheadrightarrow Hom(B, \mathcal{U}(A)).$$

2. *For each* B *and* B' *objects of* C_2*,*

$$Hom(F(B), F(B')) \approx Hom(B, \mathcal{U}(F(B')))$$

Property 2.7 *If* F *is left-adjunct to* $\mathcal{U}$*, then* F *is left semi-adjunct to* $\mathcal{U}$ *.*

Property 2.8 *If* $F : C \to C'$ *is left semi-adjunct to* $\mathcal{U} : C' \to C$*, and* C *has an initial object* $\mathcal{O}$*, then* $F(\mathcal{O})$ *is semi-initial in* C'

Proof : There are two points in the demonstration:

- Let M be an object of C'. There is a natural surjection from $Hom(F(\mathcal{O}), \mathcal{M})$ to $Hom(\mathcal{O}, \mathcal{U}(M))$. Since $\mathcal{O}$ is initial, $Hom(\mathcal{O}, \mathcal{U}(M))$ is non-empty, so there is at least one morphism $\mu : F(\mathcal{O}) \to \mathcal{M}$.
- Since $Hom(F(\mathcal{O}), F(\mathcal{O})) \approx Hom(\mathcal{O}, \mathcal{U}(\mathcal{F}(\mathcal{O})))$ and $\mathcal{O}$ is initial in C, then

$$Hom(F(\mathcal{O}), F(\mathcal{O})) = \{Id_{F(\mathcal{O})}\}$$

Then we deduce that $F(\mathcal{O})$ is semi-initial in C'. □

We can now use this notion of semi-adjunction to (meta-)define terms with variables.

Definition 2.9 *Let* Σ *be a signature,* $|\_|_\Sigma : Mod(\Sigma) \to Carr(\Sigma)$ *the functor from models of* Σ *to their carriers*[3]*, If* $|\_|_\Sigma$ *has a left semi-adjunct* $\mathcal{T}_\Sigma$*, then we say that* $\mathcal{T}_\Sigma(V)$ *is the model of terms over* V*.*

Remark 2.10 *From this definition and Proposition 2.8, we can conclude that if* $Carr(\Sigma)$ *has an initial object* $\emptyset$*, and the functor* $\mathcal{T}_\Sigma$ *exists, then* $\mathcal{T}_\Sigma(\emptyset)$ *is semi-initial in* $Mod(\Sigma)$*, hence* $\mathcal{T}_\Sigma(\emptyset) = T_\Sigma$*.*

[3]Remember that the "sets of variables" are in $Carr(\Sigma)$

3 First Properties and Applications

3.1 Sensible Signatures

A sensible signature is, in the usual formalisms, a signature such that there are ground terms of each sort. In other words, each variable can be instantiated by a ground term. In our setting this leads to the following definition:

Definition 3.1 *A* sensible *signature is a signature* Σ *such that for each* V *in* $Carr(\Sigma)$*, there is a morphism* $V \to |T_\Sigma|$ *in* $Carr(\Sigma)$*.*

3.2 Variables and Terms

We have $Hom(\mathcal{T}_\Sigma(V), \mathcal{T}_\Sigma(V)) \approx Hom(V, |\mathcal{T}_\Sigma(V)|)$, hence we have a *unit of semi-adjunction* which gives us a (canonical) morphism $i_V : V \to |\mathcal{T}_\Sigma(V)|$. One could think it modelizes the inclusion of V in $\mathcal{T}_\Sigma(V)$, but this is not always true. Consider a theory where, for each signature Σ, $Mod(\Sigma) = \{Triv_\Sigma\}$ (i.e. the algebra in which each sort has only one value), $Carr(\Sigma)$ is the category of all heterogeneous sets. Then we can define $\mathcal{T}_\Sigma(V)$ as $Triv_\Sigma$, and $V \not\subset |\mathcal{T}_\Sigma(V)|$, this is however a very rare case, and happens when Mod(Σ) is "poorer" than $Carr(\Sigma)$.

3.3 Evaluations

Let us consider $\mathcal{T}_\Sigma(|A|)$, where A is a model of Σ. We will say that a morphism from $\mathcal{T}_\Sigma(|A|)$ to A is an *evaluation* over A, if it preserves the values in A. Let $i_A : A \to \mathcal{T}_\Sigma(|A|)$ be the unit of semi-adjunction, then we can define $\mathcal{E}val_A = \{\mu : \mathcal{T}_\Sigma(|A|) \to A \setminus |\mu| \circ i_A = Id_{|A|}\}$.

$\mathcal{E}val_A$ is the set of evaluations and the evaluations are the sections of i_A. Notice that if $\mathcal{E}val_A$ is non-empty, then $|i_A|$ is a mono-morphism (i.e., $|A| \subset |\mathcal{T}_\Sigma(|A|)|$).

Lemma 3.2 *If* Σ *is a sensible signature then, for each "set of variables"* V *in* $Carr(\Sigma)$*, there is a unique evaluation morphism* $e_V : \mathcal{T}_\Sigma(\mathcal{T}_\Sigma(V)) \to \mathcal{T}_\Sigma(V)$*.*

We need the following technical lemma in order to prove the property:

Lemma 3.3 *If* Σ *is sensible, then for each* V *in* $Carr(\Sigma)$ *and each* M *in* $Mod(\Sigma)$*, there is a morphism* $\nu : \mathcal{T}_\Sigma(V) \to M$*.*

Proof : Since Σ is sensible, there is a morphism $\phi : V \to |T_\Sigma|$. Since $\mathcal{T}_\Sigma$ is left semi-adjunct to $|\_|$, we have $Hom(V, |T_\Sigma|) \approx Hom(\mathcal{T}_\Sigma(V), T_\Sigma)$, so there is a morphism $\psi : \mathcal{T}_\Sigma(V) \to T_\Sigma$. Since T_Σ is semi-initial there is a morphism $\mu : T_\Sigma \to M$, so by composition, there is a morphism $\nu : \mathcal{T}_\Sigma(V) \to M$. □

Note 3.4 *From now on, in this proof, we will note* T *for* $\mathcal{T}_\Sigma(V)$*.*

Proof of Lemma 3.2: Let us remind what semi-adjunction means (see Fig. 2): $Hom(F(A), B) \approx Hom(A, U(B))$ means: $\exists \mu = (\mu_{AB})_{AB \in \alpha \times \beta}$ a family of isomorphisms such that:

$$\begin{array}{l} \forall f{:}A' \to A \\ \forall g{:}B \to B' \\ \forall \iota{:}A \to B \\ \qquad \mu_{A'B'}(F(f); \iota; g) = f; \mu_{AB}(\iota); U(g) \end{array}$$

In our particular case, if $A = |T|$, $B = \mathcal{T}_\Sigma(T)$, $U = |\_|_\Sigma$ and $F = \mathcal{T}_\Sigma$, $\iota = Id_B$, $A' = V$ and $B' = T$, we get[4]

[4] g exists from Lemma 3.3

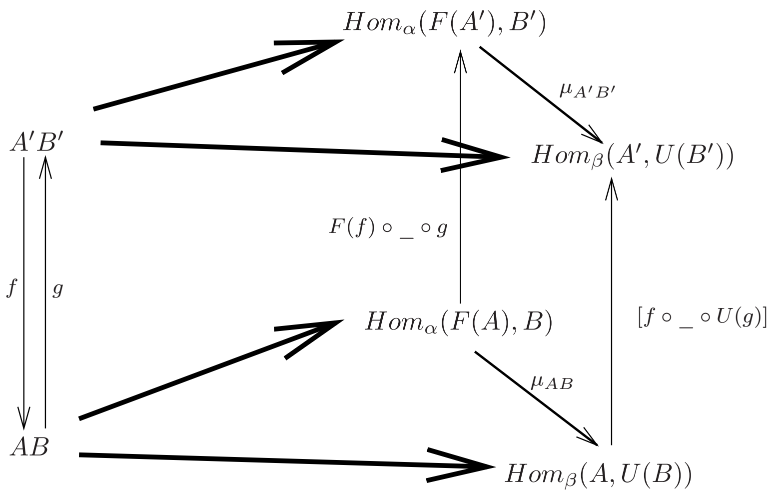

Figure 2: *Hom* isomorphism

$$\begin{aligned}&\forall f{:}V \to |T|\\ &\forall g{:}\mathcal{T}_\Sigma(T) \to T\\ &\qquad \mu_{VT}(\mathcal{T}_\Sigma(f); g) = f; i_B; |g|\end{aligned}$$

Then, if g is an evaluation $\mathcal{T}_\Sigma(T) \to T$, (i.e. $i_B; |g| = Id$), we can deduce that $f = \mu_{VT}(\mathcal{T}_\Sigma(f); g)$, and if $f = Id_{|T|}$, we can deduce that $g = \mu^{-1}_{|T|T}(Id_{|T|})$, which gives the unicity of the evaluation as the semi-adjunction co–unit, and since this co–unit always exist, it also give its existence.

3.4 Substitutions

Definition 3.5 *Given a (carrier) morphism $I : V \to |B|$, which represents the instantiation of the variables of V by values in B, since $Hom(V, |\mathcal{T}_\Sigma(B)|) \approx Hom(\mathcal{T}_\Sigma(V), \mathcal{T}_\Sigma(B))$ (from the semi-adjunction), there is a one to one correspondence between I and a morphism $\theta_I : \mathcal{T}_\Sigma(V) \to \mathcal{T}_\Sigma(B)$ which is the corresponding* substitution *morphism.*

Let us remark that the previous definition only describes a kind of "syntactical" substitution, i.e. a substitution whose only effect is to replace the variables in the term by their value, giving a morphism from $\mathcal{T}_\Sigma(V)$ to $\mathcal{T}_\Sigma(B)$. However it is always possible to compose it with an evaluation morphism to get a more usual substitution from $\mathcal{T}_\Sigma(V)$ to B.

4 Initiality Result[5]

Having defined the terms, we can now use them for some "standard" demonstrations, like the existence of an initial model among those satisfying a specification. However a part of this demonstration strongly depends on the intimate structure of the formalism and cannot be made at our high level of abstraction. Thereafter we will present a meta-result which is applicable only to formalisms that satisfies a strong property, which we define below as the *smallest congruences* property. Even if this requirement seems to be strong (it may demand a long demonstration to the author of a specification formalism), practically we have seen that it cuts the proof by the half (it is the case for ADJ, label algebras [BLGA94], . . .)

We suppose that the models of a signature Σ satisfying a specification Sp form a sub-category of Mod(Σ) that we will note Mod(Sp).

[5]This result was not presented in Santa Margherita

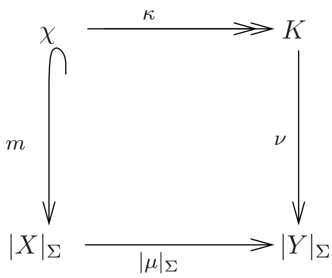

Figure 3: Commutation

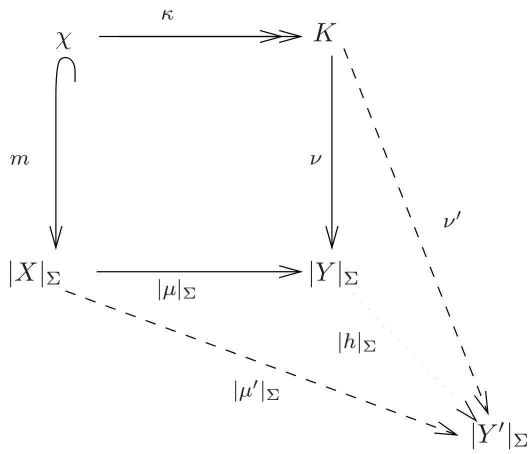

Figure 4: Minimality

4.1 Smallest Congruences

Definition 4.1 *Let Σ be a signature. Let Sp be a specification built over Σ. Let $\chi \in Carr(\Sigma)$ and $X \in Mod(\Sigma)$. Let $m : \chi \to |X|$ be a monomorphism and $\kappa : \chi \to K$ an epimorphism in $Carr(\Sigma)$. We say that Σ has* smallest congruences *if for each choice of Sp, X, χ, m and κ, there is a model Y in Mod(Sp), a morphism $\mu : X \to Y$ in Mod(Σ) and a morphism $\nu : K \to |Y|$ in $Carr(\Sigma)$ verifying that the diagram 3 commutes, that for all Y' in Mod(Sp), $\mu' : X \to Y'$ in Mod(Σ) and $\nu' : K \to |Y'|$ making a similar commutative diagram, there is a unique $h : Y \to Y'$ in Mod(Sp) such that the diagram 4 commutes, and that if Y together with morphisms μ' and ν' makes a commutative diagram such as in Fig. 4, then $\mu = \mu'$ and $\nu = \nu'$.*

Intuitively, the kernel of μ can be considered as the smallest congruence generated by the kernel of κ.

If every signature Σ has smallest congruences, then we say that the pre-institution has smallest congruences

Remark 4.2 *We can remark that the smallest congruences are very similar to pushouts, the only difference coming from restriction on the morphisms involved.*

4.2 Initial Model

Let us suppose that we are in a deterministic framework (i.e. T_Σ is initial in Mod(Σ)). Let us take $X = T_\Sigma$, $\chi = K = |T_\Sigma|$ and $m = \kappa = Id_{|T_\Sigma|}$. Let us consider now the model Y given by the smallest congruence. This model is initial in Mod(Sp).

Proof : Let us consider Y' in Mod(Sp), since Mod(Sp) is a sub-category of Mod(Σ), there is a unique morphism from T_Σ to Y', so the smallest congruence says there is a unique morphism $h : Y \to Y'$ in Mod(Sp), thus Y is initial in Mod(Sp). □

4.3 Semi-initial Model

Similarly, in non deterministic cases (i.e. if T_Σ is semi-initial), there is a semi-initial model in Mod(Sp) under the same conditions:

Since T_Σ is semi-initial, for each Y' there is at least one morphism from T_Σ to Y', so the smallest congruences property ensures there is at least one morphism from Y to Y'

4.4 Left adjoint to the forgetful functor

The "smallest congruence" property is a bit sophisticated to only establish a simple meta-result such as the existence of an initial model. As the reader may have guessed, this property has been designed in order to establish more powerful meta-results: mainly the existence of a left-adjoint to the forgetful functor.

More precisely, in a deterministic pre-institution with carriers where the $\mathcal{E}val$ sets are not empty, if for each signature morphism $\rho : \Sigma_1 \to \Sigma_2$, $Carr(\rho)$ has a left adjunct and if the "smallest congruence" property is satisfied, then the forgetful functors in Mod(Sp) have left-adjuncts.

Unfortunately, the demonstration of this meta-result [Dav] is several pages long and needs a lot of intermediate concepts which we cannot expose here for lack of space.

Conclusion

Our goal is to propose a useful meta-formalism which is a compromise between

- "enough generality": the meta-formalism should cover (almost) all the specification formalisms which are known up to now, and the concepts used by the meta-formalism should be sufficiently abstract and simple to establish understandable and general results
- "low effort to instantiate": from our experiments [Dav92], it is sometimes difficult to instantiate in a natural way a meta-formalism by some given specification theory. For example it may be necessary to extract from "what anybody would consider as the natural set of axioms" some formulas (of a given form) and to artificially put them into the signature, in order to cope with the meta-definitions. We believe that the author of a specification formalism should not have to deeply rearrange his (her) concepts in order to apply the meta-results (even if the price to pay is a little loss of generality in the definition of our meta-formalism). If the "rearrangement effort" is comparable to the effort of a direct proof, then the meta-results will not be used.

Among the general "natural" concepts which are (almost) always used by the authors of an algebraic specification formalism, the notion of carrier is very useful. In this article, we concentrated our meta-results around the notion of term and we showed that a category of carriers is suitable; it allows in particular to easily define a meta-notion of set of variables.

Moreover, we have introduced semi-initiality and semi-adjunction and used them to give the meta-definitions of the model of ground term and the model of terms with variables (with respect to a given "set of variables"). The advantage of using semi-initiality and semi-adjunction instead of initiality and adjunction is to take into account specification theories with non-determinism in a natural way.

Of course a meta-formalism is useful only if it has a corresponding "toolbox" of meta-results. We have outlined several such "tools" (meta-results) that rely on our meta-notion of terms: term evaluations, sensible signatures, substitutions, initiality and adjunction results for free[6] if smallest congruences exist.

Future works will be dedicated to the enrichment of the "toolbox", for example:

- to study the case where terms do not directly form a model
- to define a meta-notion for subterms
- to investigate how a meta-notion of term could be used in order to make more precise the structure of formulas. The form of a formula is never addressed in institutions, pre-institutions or specification frames; nevertheless logical formulas are almost always build following meta-rules such as: build atoms from terms and predicates, add connectives, add quantifiers, define positive conditional formulas (e.g. Horn clauses), etc. An example where the form of a formula can play a central role is the Birkhoff's completeness proof for positive conditional formulas... is it possible to propose a meta-version of such a result? For the moment this article is far from answering the question !

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[6] without pun

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