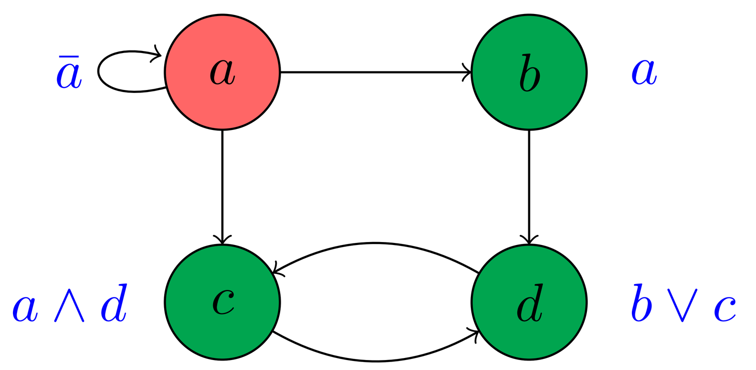
The cost of simulating a parallel BAN by a sequential one

CANA (Natural Computation), **LIF** UMR CNRS 7279, **Aix-Marseille University**, France

Aix*Marseille université

LABORATOIRE D'INFORMATIQUE FONDAMENTALE de Marseille

Boolean automata networks (BAN)

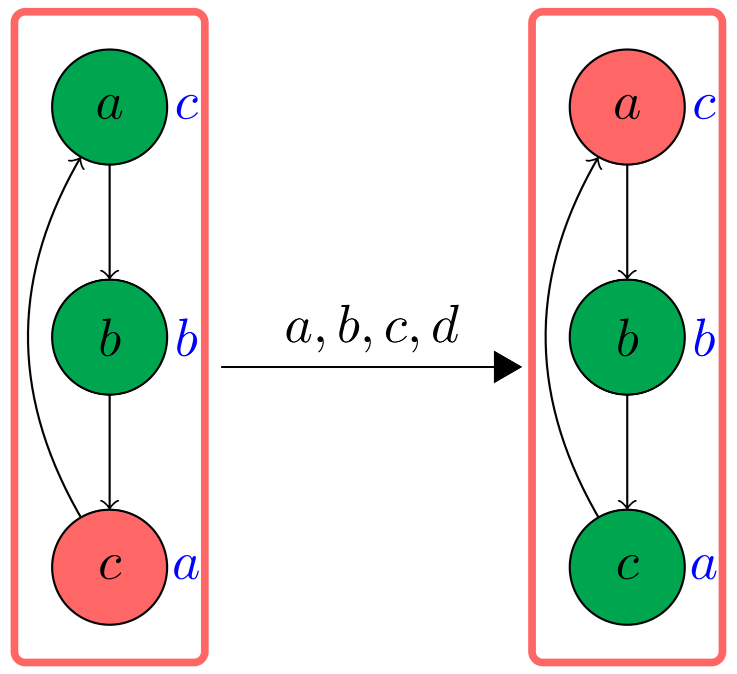


A BAN is composed of:

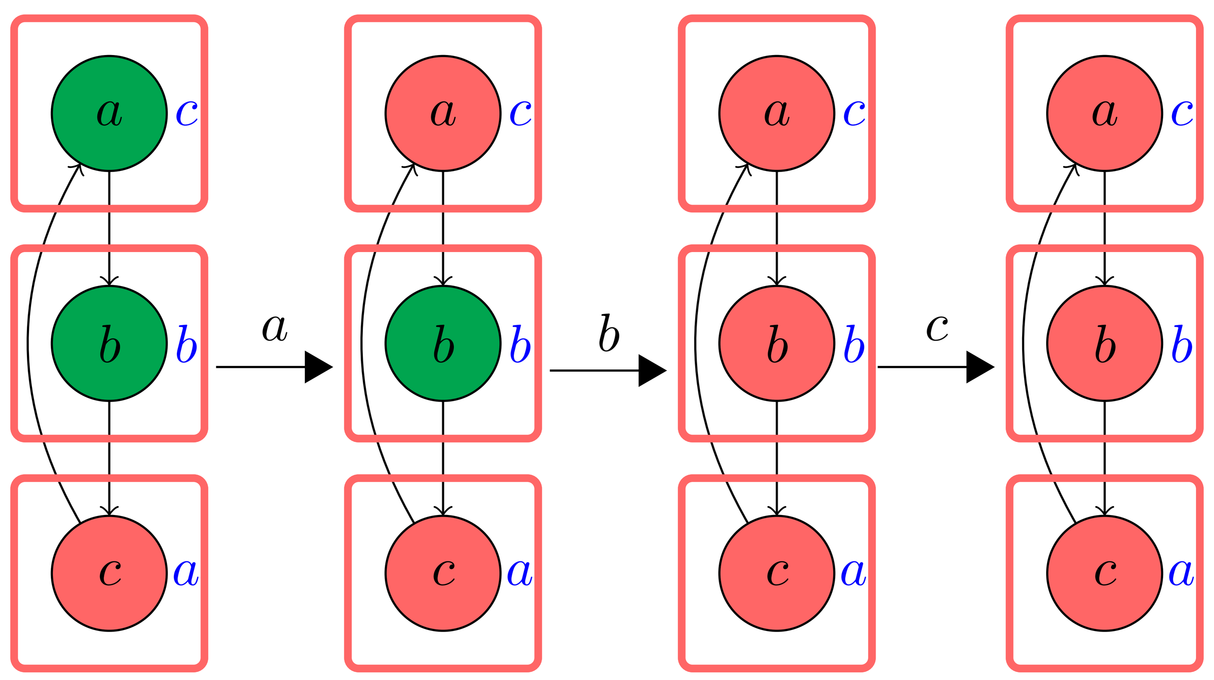
- A directed graph (called **interaction graph**)
- And for each node of the graph:
  - a Boolean value (True or False)
  - an update function which depends on the incoming neighbors of the node

Update schedule

To study the dynamics (i.e. the way it changes with time) of a BAN, we need to define its update schedule: the order of update of its nodes. All nodes updated together is a **parallel update schedule**.

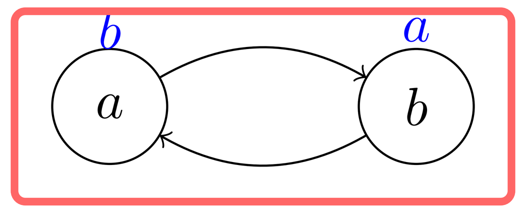


Nodes updated one at the time is a **sequential update schedule**.

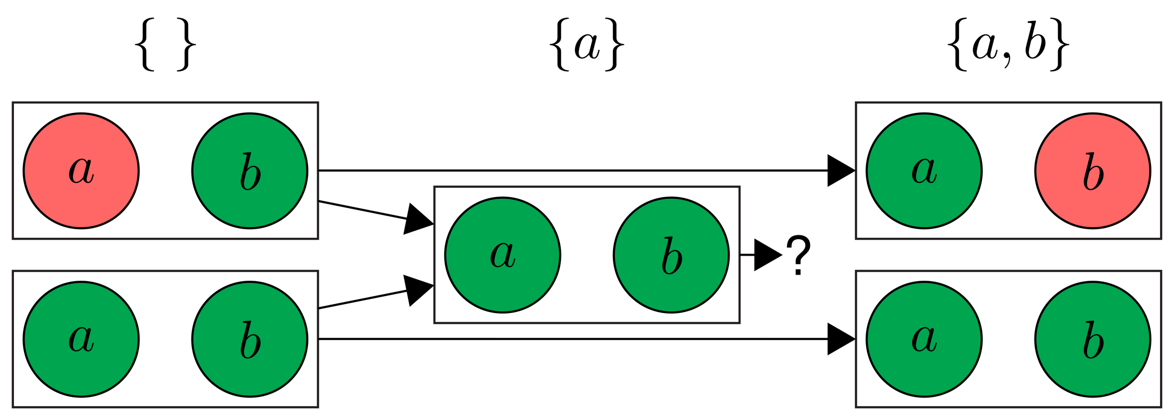


Problem

Here is a parallel BAN with no equivalent sequential BAN:

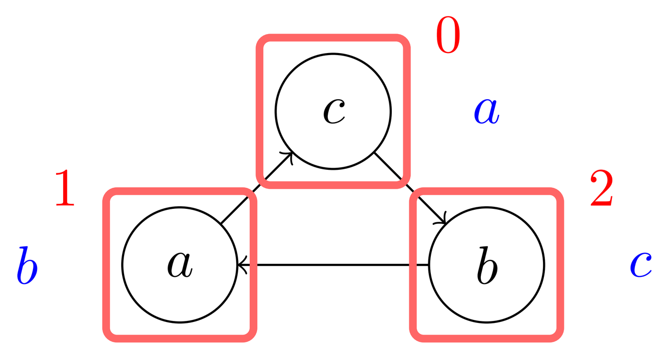


For the sake of contradiction, let us say that there is a sequential BAN with the same behavior. Let us consider the evolution of two configurations (False , True) and (True , True) :



After updating a, we have the same configuration (True , True): we have erased its previous value. However, after updating b, we should have two different configurations depending on the value of a we just destroyed. That is absurd.

Thus, if we want a sequential BAN N' with the same dynamics on a and b than N, we need an additional automaton c **updated before** a and b to save the value of a before erasing it:



Thus, we are asking two questions:

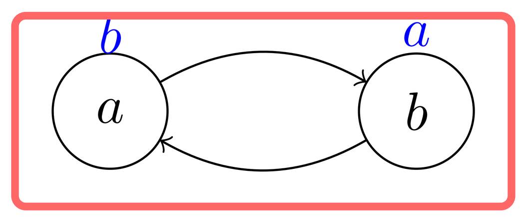
- If we have a parallel BAN N, what is the number κ_N of additional nodes of the smallest sequential BAN which simulates it?
- And for each $n \in \mathbb{N}$, what is the biggest κ_N for all BAN N of size n?

Confusion graph

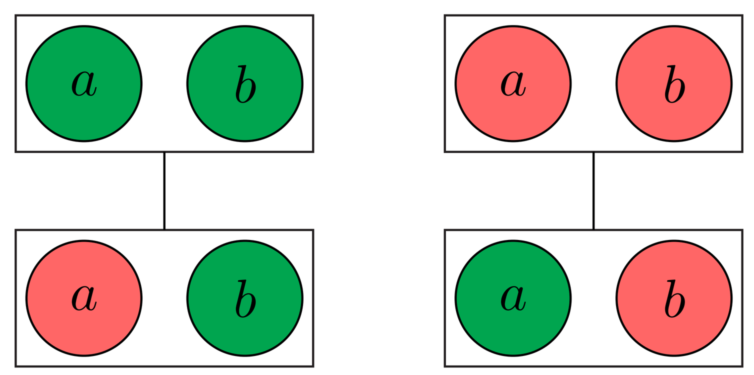
The confusion graph of a parallel BAN N has 2^n nodes: one per possible configuration.
And we have an edge between two configurations x and x' if:

- x and x' are identical when we update their first i nodes, for some $i < n$;
- x and x' are different when we update all of their nodes.

For example with this parallel BAN N:



The confusion graph G_N will be as bellow:



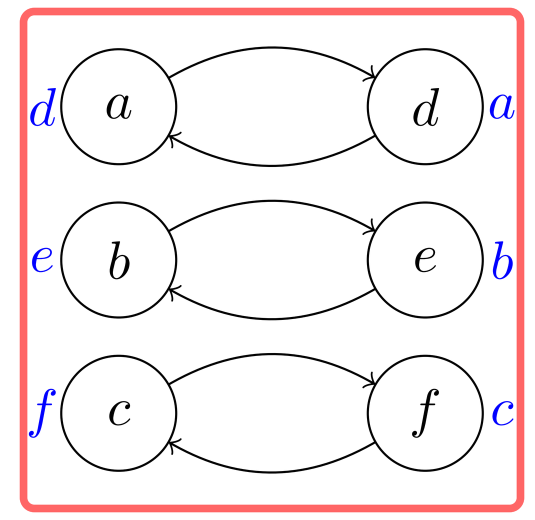
Let us consider the sequential BAN N' which simulates N. If two configurations x and x' are neighbors in the confusion graph, then the additional automata of N' have to take different values. Thus $\kappa(N) \geq \lceil log_2(\chi(G_N)) \rceil$.

Conversely, if we have a valid coloration of the confusion graph, we can build a BAN N' which simulates N using only $\lceil log_2(\chi(G_N)) \rceil$ additional automata.

Theorem. $\kappa(N) = \lceil log_2(\chi(G_N)) \rceil$

Lower Bound for κ_n

We can easily create parallel BAN a N of size n such that $\kappa(N) = n/2$. For example:



Let us consider the set X of configuration where all automata of second column are false.
X is a clique of size $2^{n/2}$ and the chromatic number of the confusion graph is at least $2^{n/2}$. As a result, $\kappa(N) \geq n/2$.

Theorem. $\kappa_n \geq n/2$.

Upper Bound of κ_n

We can prove that: $\kappa_n \leq 2n/3+2$. In the confusion graph:

1. group configurations with same image
2. sort groups by decreasing degree
3. color groups using a greedy algorithm

We can prove that using this method, we never use more than $2n/3 + 2$ colors.

Theorem. $\kappa_n \leq 2n/3 + 2$.

More

- We conjecture that we have $\kappa_n = n/2$.
- If N is a bijective BAN, then $\kappa(N) \leq \lceil n/2 \rceil$.

References & Acknowledgement

Ref. On the cost of simulating a parallel Boolean automata network with a block-sequential one https://hal.archives-ouvertes.fr/hal-01479439

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