Linear-time Constant-ratio Approximation Algorithm and Tight Bounds for the Contiguity of Cographs

Christophe Crespelle[1] and Philippe Gambette[2]

Abstract

In this paper we consider a graph parameter called *contiguity* which aims at encoding a graph by a linear ordering of its vertices. We prove that the contiguity of cographs is unbounded but is always dominated by $O(\log n)$, where n is the number of vertices of the graph. And we prove that this bound is tight in the sense that there exists a family of cographs on n vertices whose contiguity is $\Omega(\log n)$. In addition to these results on the worst-case contiguity of cographs, we design a linear-time constant-ratio approximation algorithm for computing the contiguity of an arbitrary cograph, which constitutes our main result. As a by-product of our proofs, we obtain a min-max theorem, which is worth of interest in itself, stating equality between the rank of a tree and the minimum height its path partitions.

Introduction

In many contexts, such as genomics, biology, physics, linguistics, computer science and transportation for examples, industrials and academics are led to algorithmically treat large dataset organised in the form of networks or graphs. The algorithms used to do so generally make extensive use of *neighborhood queries*, which, given a vertex x of a graph G, ask for the list of neighbors of x in G. Therefore, as pointed out by [12], due to the huge size of the graphs considered, finding compact representations of a graph providing optimal-time neighborhood queries is a crucial issue in practice.

One possible way to achieve this goal is to find an order σ on the vertices of G such that the neighborhood of each vertex x of G is an interval in σ. In this way, one can store the list of vertices of the graph in the order defined by σ and assign two pointers to each vertex: one toward its first neighbor in σ and one toward its last neighbor in σ. Therefore, one can answer adjacency queries on vertex x simply by listing the vertices appearing in σ between its first and last pointer. It must be clear that such an order on the vertices of G does not exist for all graphs G. Nevertheless, this idea turns out to be quite efficient in practice and some compression techniques are precisely

[1]Université Claude Bernard Lyon 1, DNET/INRIA, LIP UMR CNRS 5668, ENS de Lyon, Université de Lyon – `christophe.crespelle@inria.fr`

[2]Université Paris-Est, LIGM UMR CNRS 8049, Université Paris-Est Marne-la-Vallée, 5 boulevard Descartes, 77420 Champs-sur-Marne, France – `philippe.gambette@univ-mlv.fr`

based on it [2, 3]: they try to find orders of the vertices that group the neighborhoods together, as much as possible.

When one relaxes the constraints of the initial problem by rather asking for the minimum k such that there exists an order σ on the vertices of G where the neighborhood of each vertex is split in at most k intervals, this gives rise to a graph parameter called the *contiguity* of G [5]. This parameter was originally introduced in the broader context of binary matrices under the name k-consecutive-ones property [8]. It is worth to note that there are two variants of the parameter, respectively called *open contiguity* and *closed contiguity*, depending on whether one considers open neighborhoods (excluding the vertex itself) or closed neighborhoods (always containing the considered vertex). But this distinction is not fundamental as the two parameters always differ by at most one.

Here, we are interested in determining what is the worst-case contiguity for the cographs on n vertices, which are the graphs having no induced P_4 (path on 4 vertices). We also design an approximation algorithm that computes the contiguity of any cograph up to a constant ratio, in linear time with regard to the size of the input.

Related works.

Only very little is known about the contiguity of graphs. Only the class of graphs having open contiguity 1 and the class of graphs having closed contiguity 1 have been characterized: the former are biconvex graphs [4] and the latter are proper interval graphs [11]. But the classes of graphs having contiguity at most k, where k is an integer greater than 1, have not been characterized, even for $k = 2$. Actually, closed contiguity has initially been studied in the context of $0 - 1$ matrices and [8, 9, 13] showed that deciding whether an arbitrary graph has closed contiguity at most k is NP-complete for any fixed $k \geq 2$. For arbitrary graphs again, [7] (Corollary 3.4) gave an upper bound on the value of closed contiguity which is $n/4 + O(\sqrt{n \log n})$, whose interest lies in the constant $1/4$ since it is clear that the contiguity of a graph is always less than $n/2$ (where n is the number of vertices of the graph). Finally, let us mention that [5] showed that the contiguity is unbounded for interval graphs as well as for permutation graphs, and that it can be up to $\Omega(\log n/ \log \log n)$.

Our results.

In this paper, we show that even for cographs, the contiguity is unbounded, but is dominated by $O(\log n)$ for a cograph on n vertices. To this purpose, we show (in Section 4) that the contiguity of a cograph G is mathematically equivalent the maximum height of a complete binary tree included (as a minor) in the cotree of G. This also allows us to exhibit a family of cographs

$(G_n)_{n\in\mathbb{N}}$ on n vertices whose asymptotic contiguity is $\Omega(\log n)$, which implies that our $O(\log n)$ bound is tight. Furthermore, from this result, we derive in Section 5 a constant-ratio approximation algorithm that computes the contiguity of an arbitrary cograph (up to a multiplicative constant) in linear time wrt. the size of the input, that is $O(n)$ time provided that the input cograph is given by its cotree (see Section 1 for a definition). In addition, our algorithm can also provide a linear ordering σ of the vertices of G, together with the pointers from each vertex to the at most k intervals partitioning its neighborhood, that realizes a k which is in a constant ratio from the optimal one, i.e. the contiguity of G. In this case, the complexity of our algorithm is linear wrt. the size of the output, that is $O(kn)$ time.

As a by-product of our proofs, we also establish in Section 2 a min-max theorem which is worth of interest in itself: the maximum height of a complete binary tree included (as a minor) in a tree T (known as the *rank* of tree T [1, 6]) is equal to the minimum height of a partition of T into vertex-disjoint paths.

1 Preliminaries.

All graphs considered here are finite, undirected, simple and loopless. In the following, G denotes a graph, V (or $V(G)$ to avoid ambiguity) denotes its vertex set and E (or $E(G)$) its edge set. We use the notation $G = (V, E)$ and we denote $|V| = n$. The set of subsets of V is denoted by 2^V. An edge between vertices x and y will be arbitrarily denoted by xy or yx. The (open) neighborhood of x is denoted by $N(x)$ (or $N_G(x)$ to avoid ambiguity) and its closed neighborhood by $N[x] = N(x) \cup \{x\}$. The subgraph of G induced by the subset of vertices $X \subseteq V$ is denoted by $G[X] = (X, \{xy \in E \mid x, y \in X\})$.

Cographs are the graphs that do not have any P_4 (path on 4 vertices) as induced subgraph. They are also known to be the graphs G admitting a *cotree*, i.e. a rooted tree T whose leaves are the vertices of G, and whose internal nodes are labelled *series* or *parallel* with the following property: any two vertices x and y of G are adjacent iff the least common ancestor u of leaves x and y in T is a series node. Otherwise, if u is a parallel node, x and y are not adjacent.

For a rooted tree T and a node $u \in T$, the depth of u in T is the number of edges in the path from the root to u (the root has depth 0). The *height* of T, denoted by $\mathit{height}(T)$ or simply $h(T)$, is the greatest depth of its leaves. For a rooted tree T, the subtree of T rooted at u, denoted by T_u, is the tree induced by node u and all its descendants in T. In the following, the notion of *minors* of rooted trees is central. This is a special case of minors of graphs (see e.g. [10]), for which we give a simplified definition in the context of rooted trees.

Definition 1 *The* contraction of edge *uv in a rooted tree T, where u is the parent of v, consists in removing v from T and assigning its children (if any) to node u.*
A rooted tree T' is a minor *of a rooted tree T if it can be obtained from T by a sequence of edge contractions.*

Let us now formally define the contiguity of a graph.

Definition 2 *A* closed p-interval-model *(resp. open p-interval-model) of a graph $G = (V, E)$ is a linear order σ on V such that $\forall v \in V, \exists (I_1, \ldots, I_p) \in (2^V)^p$ such that $\forall i \in [\![1, p]\!]$, I_i is an interval of σ and $N[v] = \bigcup_{1 \leq i \leq p} I_i$ (resp. $N(v) = \bigcup_{1 \leq i \leq p} I_i$).*
The closed contiguity *(resp.* open contiguity*) of G, denoted by $cc(G)$ (resp. $oc(G)$) , is the minimum integer p such that there exists a closed p-interval-model (resp. open p-interval-model) of G.*

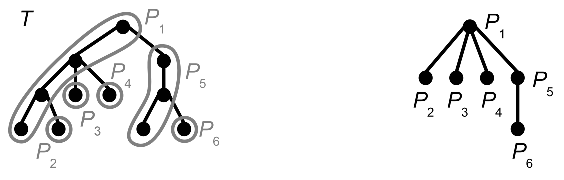
It is worth to note that the closed and open contiguity never differ by more than one. Indeed, given a closed k-interval model of a graph G, we directly get an open $k+1$-interval model for G by simply splitting, for each vertex x, the interval containing x into (at most) two intervals. Conversely, adding (at most) one trivial interval $\{x\}$ for each vertex x in an open k-interval-model results in a closed $(k+1)$-interval model. Therefore, in all the rest of the paper, we only consider closed contiguity, but all our results also hold for open contiguity. In the following, we abusively extend the notion of contiguity to cotrees referring to the contiguity of their associated cograph.

2 Some general results on the rank of trees

In this section, we give two general results on trees which will play a key role in the rest of the paper. The first result links the rank of a tree T with the minimum height of a partition of vertices of T into paths.

The *rank* [1, 6] of a tree T is the maximal height of a complete binary tree obtained from T by edge contractions, that is: $rank(T) = \max\{h(T') \mid T'$ complete binary tree, minor of $T\}$. A *path partition* of a tree T is a partition $\{P_1, \ldots, P_k\}$ of $V(T)$ such that for any i, the subgraph $T[P_i]$ of T induced by P_i is a path, as shown in Figure 1(a). The *partition tree* of a path partition $\mathcal{P}$, denoted by $T_p(\mathcal{P})$, is the tree whose nodes are P_i's and where the node of $T_p(\mathcal{P})$ corresponding to P_i is the parent of the node corresponding to P_j iff some node of P_i is the parent in T of the root of P_j (see Fig. 1(b)). The height of a path partition $\mathcal{P}$ of a tree T, denoted by $h(\mathcal{P})$, is the height $h(T_p(\mathcal{P}))$ of its partition tree. The *path-height* of T is the minimal height of a path partition of T, that is $ph(T) = \min\{h(\mathcal{P}) \mid \mathcal{P}$ path partition of $T\}$.

Theorem 1 *For any rooted tree T, $rank(T) = ph(T)$.*



(a) (b)

Figure 1: A tree T and a path partition $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ of T (a), as well as the partition tree of $\mathcal{P}$ (b).

Sketch of proof. It is not difficult to show that the path-height of a tree T is at least the path-height of any tree T' included in T as a minor. On the other hand, a simple recursion on the height of a complete binary tree shows that its path-height is at least its height. It follows that the path-height of any tree T is at least the maximum height of a complete binary tree T' included as a minor in T, that is $ph(T) \geq rank(T)$. The converse inequality, namely $ph(T) \leq rank(T)$, can be shown by induction on $rank(T)$. Indeed, consider a tree T such that $rank(T) = k + 1$. The nodes u of T such that $rank(T_u) = k + 1$ form a path P containing the root of T. By definition, any node $v \notin P$ is such that $rank(T_v) \leq k$. Then, by the induction hypothesis, $ph(T_v) \leq rank(T_v) \leq k$. And it follows that $ph(T) \leq max_{v \notin P} ph(T_v) + 1 \leq k + 1 = rank(T)$. A detailed version of this proof is given in Appendix A, in Theorem **??**. □

We now consider *bicolored trees*, i.e. trees whose nodes are colored either black or white. We define the *black rank* (resp. *white rank*), denoted $r_B(T)$ (resp. $r_W(t)$), of a bicolored tree T as the maximum height of an entirely black (resp. entirely white) complete binary tree being a minor of T.

Theorem 2 *For any bicolored complete binary tree T, $r_B(T) + r_W(T) \geq h(T) - 1$.*

Sketch of proof. The proof is by induction on $h(T)$. Consider a complete binary bicolored tree T of height $k + 1$, whose root is colored black wlog. and has two children denoted by u_1 and u_2. If $r_B(T_{u_1}) = r_B(T_{u_2})$, then $r_B(T) = r_B(T_{u_1}) + 1$. And since $r_W(T) \geq r_W(T_{u_1})$, by the induction hypothesis, we obtain $r_B(T) + r_W(T) \geq h(T_{u_1}) + 1 - 1 = h(T) - 1$. On the other hand, if $r_B(T_{u_1}) \neq r_B(T_{u_2})$, then assume wlog. that $r_B(T_{u_1}) > r_B(T_{u_2})$. Then, we have $r_B(T) + r_W(T) \geq r_B(T_{u_1}) + r_W(T_{u_2}) \geq r_B(T_{u_2}) + 1 + r_W(T_{u_2})$, and by the induction hypothesis, we obtain the desired inequality for T. A detailed version of this proof is given in Appendix A, in Theorem **??**. □

3 An upper bound for the contiguity of cographs

We now prove that the contiguity of any cograph is linearly bounded by the rank of its cotree T (Theorem 3 below) by using a path partition of T of minimal height h. Our proof is by induction on h and Lemma 1 below constitutes the recursive encoding step of our proof.

In a path partition of T, the path containing the root naturally induces a partition of the leaves of T, i.e. the vertices of the corresponding cograph G, as described in the following definition. A *root-path decomposition* (see Fig. 2) of a rooted tree T is a set $\{T_1, \ldots, T_p\}$ of disjoint subtrees of T, with $p \geq 2$, such that every leaf of T belongs to some T_i, with $i \in [1..p]$, and the sets of parents in T of the roots of T_i's is a path containing the root of T.

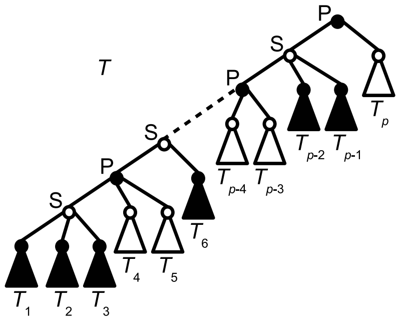


Figure 2: The root-path decomposition $\{T_1, \ldots, T_p\}$ of a rooted tree T.

Lemma 1 (Caterpillar Composition Lemma) *Given a cograph $G = (V, E)$ and a root-path decomposition $\{T_i\}_{1 \leq i \leq p}$ of its cotree, where X_i is the set of leaves of T_i, $cc(G) \leq 2 + \max_{i \in [1..p]} cc(G[X_i])$.*

Sketch of proof. It is straightforward to check that for any $i \in [1..p]$ and for any vertex $x \in X_i$, the neighbors of x that are not in X_i are split in at most two intervals in the order σ given on Fig. 3. Therefore, by choosing for σ an order on each X_i that realizes the contiguity of $G[X_i]$, we obtain that in σ, the neighborhood of any vertex $x \in X_i$ of G is split in at most $2 + cc(G[X_i])$ intervals, which proves the lemma. A detailed version of this proof is given in Appendix B, in Lemma **??**. □

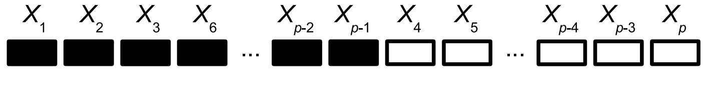


Figure 3: The general structure of the order σ used in Lemma 1 for the root-path decomposition of Fig 2.

From this, we deduce an upper bound on the contiguity of a cograph depending on the rank of its cotree.

Theorem 3 *For any cograph G having cotree T, $cc(G) \leq 2\,rank(T) + 1$.*

Sketch of proof. The proof is by induction on $rank(T)$. The inequality holds for $rank(T) = 0$. Consider a cograph G whose cotree T is of rank $k+1$, with $k \geq 0$. The nodes u of T such that $rank(T_u) = k+1$ form a path containing only internal nodes of T and containing its root. Therefore, this path induces a root-path decomposition $\{T_1, \ldots, T_p\}$ of T as shown on Fig. 2. By definition, this root path decomposition is such that for any $i \in [1..p]$, tree T_i is of rank at most k. Then, by the induction hypothesis, $cc(G[X_i]) \leq 2\,rank(T_i) + 1$. From Lemma 1, we have that $cc(G) \leq 2 + \max_{i\in[1..p]} cc(G[X_i])$. It follows that $cc(G) \leq 2 + \max_{i\in[1..p]}\{2\,rank(T_i) + 1\} \leq 2 + 2k + 1 = 2(k+1) + 1$, which ends the induction and the proof of Theorem 3. A detailed version of this proof is given in Appendix B, in Theorem **??**. □

As the rank of a tree T is bounded by the logarithm of its number of leaves, we directly obtain the following corollary.

Theorem 4 *If $G = (V, E)$ is a cograph, then $cc(G) \leq 2\log_2|V| + 1$.*

4 A lower bound for the contiguity of cographs

In this section, we show that the rank of a cotree also provides a lower bound on the contiguity of its associated cograph. Together with the result of the previous section, this give that the contiguity of a cograph and the rank of its cotree are mathematicaly equivalent functions, which is at the core of the approximation algorithm we present in next section. We also use the lower bound to exhibit a family of cographs on n vertices whose contiguity is at least $\Omega(\log n)$, showing that the upper bound of the previous section is tight.

Lemma 2 *Let G_k be the underlying undirected graph of the transitive closure of the directed rooted complete ternary tree T^k of depth $k \geq 0$. Then, we have $cc(G_k) \geq k+1$.*

Proof: We prove it by induction on k. We obviously have $cc(G_0) \geq 1$.

Now, suppose that $cc(G_k) \geq k+1$ for some $k \geq 0$, and let us show that $cc(G_{k+1}) \geq k+2$. Consider a $cc(G_{k+1})$-interval-model of G_{k+1}, and denote by σ the corresponding order on the vertices of G. We denote by r the root of T^{k+1}, and by v_1, v_2 and v_3 its three children in T^{k+1}. Since there are at most two vertices of T^{k+1} that are next to r in σ, then there exists $i \in [\![1, 3]\!]$

such that no vertex of $T_{v_i}^{k+1}$ is next to r in σ. Denoting by G_{v_i} the subgraph of G_{k+1} induced by the vertices of $T_{v_i}^{k+1}$, the induction hyptothesis gives that $cc(G_{v_i}) \geq k+1$. In particular, this implies that in the restriction of σ to the vertices of G_{v_i}, there exists some $v \in G_{v_i}$ such that its neighborhood in G_{v_i} is split into at least $k+1$ intervals. Since r is adjacent to v in G_{k+1}, and since the at most two vertices next to r in σ are not in the neighborhood of v, it follows that the neighborhood of v requires one more interval for r in σ. Thus, $cc(G_{k+1}) \geq k+2$. □

It is worth to note that G_k is a cograph. Indeed, its cotree can be recursively built as follows. The cotree of G_1 (which is also known as the *claw* graph) is made of one series root having two children: a leaf and a parallel node having three leaf children. And replacing these three leaves by three copies of the cotree of G_k, for any $k \geq 1$, results in the cotree of G_{k+1}. From now on, in order to use the result of Theorem 2, we considered cotrees bicolored as explained in the following definition. The *bicolored cotree* of a cograph G is its cotree where the parallel nodes are colored black and the series nodes are colored white.

Lemma 3 *For any cograph G, whose bicolored cotree has a white root and black rank at least $2k$, we have $cc(G) \geq k+1$.*

Sketch of proof. We prove by induction on k that G contains G_k as an induced subgraph, which has contiguity at least $k+1$ from Lemma 2. Since G has black rank at least $2k$, its cotree T contained, as a minor, an entirely black complete binary tree T'. Consider the two nodes at depth 1 in T', they have at least four different children in T. For three of them u_1, u_2 and u_3, take one leaf of their subcotree in T: since the nodes of T' are all black (i.e. parallel), this gives three independant vertices denoted x_1, x_2 and x_3. Now remind that the root of T is white (i.e. series) and must have a child which is not an ancestor of the root of T'. Take a leaf y being a descendant of this child: y, x_1, x_2 and x_3 induces the claw graph, i.e graph G_1 of Lemma 2. Finally, note that for any $u \in \{u_1, u_2, u_3\}$, u is necessarily white (as a parallel node of a cotree has only series, or leaf, children) and the cotree T_u has black rank at least $2k-2$. Then, from the induction hypothesis, the cograph associated to T_u contains G_{k-1} as an induced subgraph. And from the recursive construction of the cotree of G_k that precedes Lemma 3, we conclude that G contains G_k as an induced subgraph. □

Theorem 5 *For any cograph G and its cotree T, $cc(G) \geq (rank(T) - 7)/4$.*

Proof: Consider the bicolored cotree T of G. Then, from Lemma 2, $r_B(T) + r_W(T) \geq rank(T) - 1$, so either $r_B(T) \geq (rank(T) - 1)/2$, or $r_W(T) \geq (rank(T) - 1)/2$.

In the first case, $r_B(T) \geq 2((rank(T) - 3)/4) + 1$, so T has a black complete binary tree of height $2((rank(T)-3)/4)$, below a white node, as a minor, so Lemma 3 implies that $cc(G) \geq (rank(T)-3)/4+1 \geq (rank(T)-7)/4$.

In the second case, the bicolored cotree $T_{\bar{G}}$ of the complement of G is simply the bicolored cotree of G where series and parallel nodes, and so black and white nodes, have been exchanged. Then, we have $r_B(T_{\bar{G}}) = r_W(T) \geq (rank(T)-1)/2$. Lemma 3 implies similarly as above that $cc(\bar{G}) \geq (rank(T) - 3)/4 + 1$. We can easily check that $cc(\bar{G}) \leq cc(G) + 2$ (the closed neighborhood of any vertex x in $\bar{G}$ is composed by at most $cc(G)+1$ intervals surrounding the intervals of the closed neighborhood of x in G, plus possibly one interval for vertex x), so $cc(G) \geq (rank(T) - 3)/4 + 1 - 2 = (rank(T) - 7)/4$. □

From Theorem 5 we obtain that the contiguity of cographs is unbounded. Moreover, since the rank of a complete binary tree is, by definition, its height, it follows that the family of cographs having a complete binary cotree reaches the $O(\log n)$ upper bound of Theorem 4, showing that this bound is tight.

Corollary 1 *For any cograph G whose cotree is a complete binary tree, $cc(G) = \Omega(\log n)$.*

As a direct consequence of Theorems 5 and 3, we obtain the equivalence between the contiguity of a cograph and the rank of its cotree, which is at the core of the approximation algorithm presented in next section.

Corollary 2 *For any cograph G with cotree T, $cc(G) = \Omega(rank(T))$.*

5 An approximation algorithm for the contiguity of cographs

We now provide an algorithm that outputs an integer k and a k-interval-model of the cograph G given as input, such that k is in a constant ratio from the contiguity of G. Our algorithm takes as input a cograph G given by its cotree, which takes $O(n)$ space, and outputs a k-interval model of G in $O(kn)$ time, i.e. linear wrt. the size of the output. Alternatively, if only the value of k is needed, the algorithm runs in $O(n)$ time, which is linear wrt. the size of the input.

First step: approximation of the contiguity of G.

The first step of our algorithm computes an integer k which is in a constant ratio from the contiguity of G. To this purpose, we simply compute recursively the rank of T_u for any node $u \in T$ by a bottom-up process: for a node u we have either i) $rank(T_u) = max_{v \text{ child of } u} rank(T_v)$ if this maximum is reached by only one child of u, or ii) $rank(T_u) = 1 + max_{v \text{ child of } u} rank(T_v)$ otherwise. Then, our algorithm outputs the value $k = 2\,rank(T_r) + 1$, where r is the root of T, and so $rank(T_r) = rank(T)$. This clearly takes $O(n)$ time.

Furthermore, the approximation ratio ρ of this algorithm is constant. Using Theorem 5 and the fact that $cc(G) \geq 1$, we deduce that $\rho = (2\,rank(T) + 1)/cc(G) \leq (2\,rank(T) + 1)/max(1, (rank(T) - 7)/4)$. This function reaches its maximum of 23 for $rank(T) = 11$, and then the algorithm performs a 23-approximation of the closed contiguity (which can indeed be lower to $8 + \epsilon$, where ϵ is a positive constant as small as required, by using a constant-time precomputing step).

Second Step: building a k-interval model of G.

We follow the construction provided by Theorem 3 and Lemma 1 (see Fig. 3) in order to output a k-interval-model of G, where $k = 2\,rank(T) + 1$. During this process, we build the order σ on the vertices as well as a table $Neighborhoods$ of n tables of $2k$ pointers to the bounds of the intervals of each vertex in order σ. To this purpose, we call the recursively defined routine $Build(u)$, where u is a node of T, on the root r of T. $Build(u)$ outputs an order σ_u on the subset X_u of vertices of G being the leaves of T_u and table $Neighborhoods_u$ containing the pointers of the vertices of X_u toward σ_u. Routine $Build(u)$ proceeds in three steps as follows.
i) thanks to the ranks computed in the first step of the algorithm, $Build(u)$ finds the subset P_u of nodes $\tilde{u}$ of T_u such that $rank(T_{\tilde{u}}) = rank(T_u)$, and the subset C_u of children in T of nodes $\tilde{u} \in P_u$.
ii) $Build(u)$ recursively calls $Build(v)$ for all nodes $v \in C_u$.
iii) $Build(u)$ builds σ_u by concatenating all the orders σ_v returned by the recursive calls as shown on Fig. 3, and builds $Neighborhoods_u$ by merging all the tables $Neighborhoods_v$ returned by the recursive calls. Then, for each $v \in C_u$ and for each vertex $x \in X_v$, we add to $Neighborhoods_u$ the pointers of x toward the at most two intervals of σ_u formed by its neighbors that are not in X_v, as explained in the proof of Lemma 1.

The terminal case of Routine $Build(u)$ is when u is a leaf of T, for which the computation of σ_u and $Neighborhoods_u$ is trivial and takes constant time. The fact that a call to $Build(r)$ indeed gives the desired k-interval model of G comes from the fact that the routine follows the constructive proof of Theorem 3. Let us analyze its complexity. Step **i)**, Step **ii)** and the construction of σ_u in Step **iii)** take $O(|C_u| + |P_u|)$, that is $O(n)$ for the whole process on tree T. In Step **iii)**, the merge of table $Neighborhoods$ and the addition of the pointers to the two new intervals of each vertex take $O(X_u)$ time. It turns out that, during the whole process on T, a vertex x will be involved in at most h different sets X_u, where h is the height of the path partition of T defined by the set of paths P_u computed along the process.

From Theorem 3, $h = rank(T) = (k-1)/2$. Thus, the total computation time of the k-interval model output by our algorithm is $O(kn)$ time.

Conclusion

We showed that the contiguity of a cograph is equivalent to the maximum height of a complete binary tree contained in its cotree as a minor. From this, we obtained a tight $O(\log n)$ upper bound on the maximum contiguity of a cograph on n vertices. Even more interesting, this allowed us to design a linear time algorithm that does not only compute an approximation of the contiguity of a cograph G but also provides a k-interval model realizing a k which is in a constant-ratio to the optimal one, i.e. the contiguity of G. Then, the first question raised by our work is whether it is possible to compute efficiently the exact value of the contiguity of a cograph and to provide a model realizing this optimal value. Another key perspective is to extend our results to larger classes of graphs, such as permutation graphs (which are a proper generalization of cographs) and interval graphs. Does the $O(\log n)$ upper bound still hold for those graphs? Is it possible to compute efficiently an exact or approximated value of their contiguity?

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