Termination of the Iterated Strong-Factor Operator on Multipartite Graphs*

Christophe Crespelle[1], Thi Ha Duong Phan[2], and The Hung Tran[3]

[1] Université Claude Bernard Lyon 1, DANTE/INRIA,
LIP UMR CNRS 5668, ENS de Lyon, Université de Lyon,
christophe.crespelle@inria.fr

[2] Institute of Mathematics, Vietnam Academy of Science and Technology,
18 Hoang Quoc Viet, Hanoi, Vietnam, phanhaduong@math.ac.vn

[3] LIAFA, Université Paris-Diderot, hung.tran-the@liafa.jussieu.fr

Abstract. The clean-factor operator is a multipartite graph operator that has been introduced in the context of complex network modelling. Here, we consider a less constrained variation of the clean-factor operator, named strong-factor operator, and we prove that, as for the clean-factor operator, the iteration of the strong-factor operator always terminates, independently of the graph given as input. Obtaining termination for all graphs using minimal constraints on the definition of the operator is crucial for the modelling purposes for which the clean-factor operator has been introduced. Moreover we show that the relaxation of constraints we operate not only preserves termination but also preserves the termination time, in the sense that the strong-factor series always terminates before the clean-factor series. In addition to those results, we answer an open question from Latapy et al. [12] by showing that the iteration of the factor operator, which is a proper relaxation of both operators mentioned above, does not always terminate.

1 Introduction

One of the main challenges in modelling real-world complex networks (like internet topology, web graphs, social networks, or biological networks) is to design general models able to reproduce both the heterogeneous degree distribution of these networks and their high local density (clustering coefficient). One of the most promising approaches to do so is the one proposed by [7, 8], which aims at generating synthetic complex networks by generating their maximal cliques rather than their edges. The main difficulty in this approach is to reproduce correctly the overlaps of the maximal cliques of the graph, which is prevalent in practice.

* This work was partially supported by the PICS program of CNRS (France), by the Vietnam Institute for Advanced Study in Mathematics (VIASM) and by the Vietnam National Foundation for Science and Technology Development (NAFOSTED).

To that purpose, [12] proposes to encode the non-trivial overlaps of the maximal cliques of a graph G by a multipartite graph which is defined by iteratively applying a multipartite-graph operator, named the *weak-factor graph*, starting from the vertex-clique-incidence bipartite graph of G (see Definition 4 below and example on Figure 1). Unfortunately, the most natural definition of this operator gives series that do not terminate for some graphs G. In these cases, the object on which is based the random generation process of the model is undefined. In order to solve this issue, [12] designed a variation of the weak-factor operator, called the clean-factor, such that the corresponding series terminates for all graphs. The idea of this variation is to add some constraints to the factorising step defining the operator (see Definition 1 below) in order to force termination of the series and still capture the overlapping structure of the maximal cliques of the graph. But it turns out that the constraints added to the operator to obtain termination make the generation process of the model much more difficult to design. Therefore, for modelling purposes, it is crucial to guarantee termination for all graphs by imposing constraints as light as possible. We believe that this question of finding the minimal constraints that guarantee termination of the series is also of great theoretical interest.

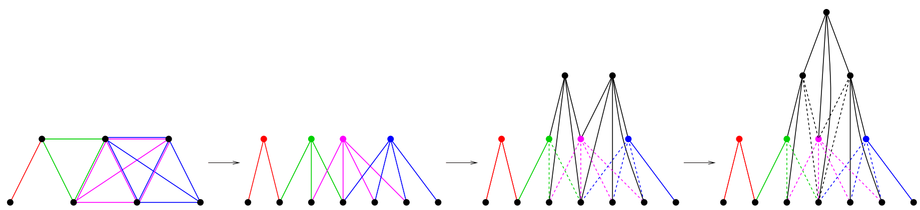

Fig. 1. Example of the weak-factor series of some graph G. From left to right: the original graph $G = G_0$, its vertex-clique-incidence bipartite graph G_1, the tripartite graph G_2 of the weak-factor series of G, and the quadripartite graph G_3 of the series. In this case, the weak-factor series terminates as the factorisation of G_3 is not effective (see Definition 1). The dashed edges are those belonging to some non-trivial maximal bicliques used in the factorisation steps.

Our contribution Our main contribution is to design a relaxation of the clean-factor operator [12], called the strong-factor operator, which is much less constrained and for which we prove that the corresponding series also terminates for all graphs. Namely, we replace the condition requiring equality of the neighbourhoods of vertices in the definition of the clean-factor operator by a condition requiring only that these vertices share at least two neighbours in common, which constitutes a strong relaxation of the previous definition. In addition, we show that this relaxation not only preserves termination but also does not delay it: the strong-factor series, though less constrained, always terminates before the clean-factor series. For sake of completeness, let us mention that in [4], which is a complete and improved version of [12], the constraints in the definition of the clean-factor operator are slightly different and are expressed in a weaker way. But these weaker constraints actually imply that the constraints of the definition used in [12] are also satisfied. Therefore, the strong-factor operator introduced here is a proper relaxation of both versions [12, 4] of the clean-factor operator.

Besides the results we obtain on the termination of the strong-factor series, we also provide a complete characterisation of the levels of the series, in terms of intervals of a poset, that is worth of interest in itself. This characterisation is very simple and gives an insight on the structure of the clean-factor series that, we believe, may also be useful to prove termination or non-termination of other multipartite graph operators. In addition, it provides an efficient way to compute the strong-factor series, by avoiding the computation of maximal bicliques.

Finally, we answer an open question of [12] by showing that the factor series, which is a relaxation of both the clean-factor series and the strong-factor series, does not terminate for some graphs.

Related work The strong-factor operator which we study here is a variation of the weak-factor operator, which operates on multipartite graphs and which is defined using the bicliques between the upper level and the rest of the multipartite graph. For graphs, closely related operators have been defined using the cliques or the bicliques of the graph, and many works addressed the question of convergence of the series obtained by iteratively applying these operators to an input graph. There exist several definitions of convergence in the literature. The notion of termination we use here for the multipartite graph series we consider is somehow equivalent to the convergence notion used in [1] in the context of graph series, and is a particular case of convergence of the definition used in [14].

For the well-known clique graph operator (see [15] for a survey) the question of convergence has received a lot of attention [14, 1]. Most of the efforts focussed on obtaining convergence results, or divergence results, for some particular graphs or graph classes [10, 9, 11, 13]. Similar questions have been addressed recently for the biclique graph operator [5, 6], which also operates on graphs but using bicliques instead of cliques. Let us mention, that another closely related graph operator called edge-clique-graph operator has been studied (see e.g. [3, 2]) but, to the best of our knowledge, the question of the convergence of its iterated series has not been investigated.

It must be clear that none of these three operators, clique graphs, biclique graphs and edge-clique graphs, which are defined on graphs, is equivalent to one of the multipartite-graph operators we consider here. And the convergence or divergence results obtained previously for these graph operators do not imply the non-termination and termination results we prove here for the factor graph and the strong-factor graph respectively.

Moreover, it is worth noticing that, even though it deals with a notion of convergence, the question we address in this paper is orthogonal, and complementary, to the one addressed in all the previously cited works. Indeed, we do not intend to characterize the graphs for which the operator we study, namely the weak-factor operator, converges or diverges. Instead, we aim at determining minimal constraints that can be imposed to this operator in order to obtain convergence for all graphs.

2 Notations and preliminary definitions

All graphs considered here are finite, undirected and simple (no loops and no multiple edges). A graph G having vertex set V and edge set E will be denoted by $G = (V, E)$. We also denote by $V(G)$ the vertex set of G. The edge between vertices x and y will be indifferently denoted by xy or yx. A clique of a graph G is a subset of its vertices that are all pairwise adjacent, and a maximal clique is a clique maximal for inclusion. We denote $\mathcal{K}(G)$ the set of maximal cliques of a graph G, and $N(x)$ the neighbourhood of a vertex x in G.

A k-partite graph G is a graph whose vertex set is partitioned into k parts, with edges between vertices of different parts only (a bipartite graph is a 2-partite graph, a tripartite graph a 3-partite graph, etc): $G = (V_0, \ldots, V_{k-1}, E)$, where the V_i's are pairwise disjoint, and with $E \subseteq \{uv \mid u \in V_i, v \in V_j, i \neq j\}$. The vertices of V_i, for any i, are called the *i-th level* of G, and the vertices of V_{k-1} are called the *upper vertices* of G.

When $G = (V_0, \ldots, V_{k-1}, E)$ is k-partite, we denote by $N_i(x)$, where $0 \leq i \leq k-1$, the set of neighbours of x at level i: $N_i(x) = N(x) \cap V_i$. A *biclique* of a graph is a set of vertices of the graph inducing a complete bipartite graph, and a maximal biclique is a biclique maximal for inclusion. We denote by $B(G)$ the vertex-clique-incidence bipartite graph of $G = (V, E)$: $B(G) = (V, \mathcal{K}(G), E')$ where $E' = \{vc \mid c \in \mathcal{K}(G),\ v \in c\}$. A *non-trivial biclique* of a bipartite graph is a biclique having at least two vertices in the upper level and at least two vertices in the bottom level. Two sets have a *non-trivial intersection* if they share at least two elements. In all the paper, we denote $\mathcal{L}$ the inclusion order of the non-trivial intersections of maximal cliques of a graph G (there will be no confusion on the graph G referred to when we use this notation).

For two non-negative integers $a, b \in \mathbb{N}$, we use the notation $[\![a, b]\!]$ for the set $\{p \in \mathbb{N} \mid a \leq p \leq b\}$, with the convention $[\![a, b]\!] = \varnothing$ if $a > b$.

In all the paper, an operation will play a key role, we name it *factorisation* and define it generically as follows.

Definition 1 (factorisation of a k-partite graph with respect to V'_k [4]). *Given a k-partite graph $G = (V_0, \ldots, V_{k-1}, E)$ with $k \geq 2$ and a set V'_k of subsets of $V(G)$, we define the factorisation of G with respect to V'_k as the $(k+1)$-partite graph $G' = (V_0, \ldots, V_k, E \cup E_+)$ where:*

- *V_k is the set of maximal (with respect to inclusion) elements of V'_k,*
- *$E_+ = \{Xy \mid X \in V_k$ and $y \in X\}$.*

When $V_k \neq \emptyset$, the factorisation of G is said to be effective*.*

A factorisation operation with respect to some set V'_k defines a multipartite graph operator, the iteration of which gives rise to a series of multipartite graphs as defined below.

Definition 2 (series of multipartite graphs associated to a factorisation operation [4]). *Given a factorisation operation that associates any k-partite graph $G = (V_0, \ldots, V_{k-1}, E)$ with $k \geq 2$ to a $k+1$-partite graph G' obtained by factorisation of G with respect to some set V'_k (see Definition 1), we define the series of multipartite graphs $(G_i)_{i \geq 1}$, associated to this factorisation operation and generated by a graph $G_0 = (V_0, E_0)$, by: $G_1 = B(G_0)$ is the vertex-clique-incidence bipartite graph of G_0 (in which the cliques are on the upper level of $B(G_0)$) and, for all $i \geq 1$, $G_{i+1} = G'_i$ when the factorisation of G_i is effective, and G_{i+1} is undefined otherwise.*

Definition 3 (termination of the series [4]). *We say that the series $(G_i)_{1 \leq i \leq n}$ associated to some factorisation operation* terminates *iff for some $i \geq 1$ the factorisation is not effective, then all subsequent graphs of the series are undefined and the series reduces to a finite sequence.*

Remark 1. Compared to the notions of convergence introduced for the iterated series of clique graphs (see e.g. [14, 1]), note that here, since the factorisation of a multipartite graph G always contains G as an induced subgraph, there are only two possible behaviours of the series: either it terminates or the number of vertices in G_i tends to infinity.

In the rest of the paper, we will refine the notion of factorisation by using different sets V'_k on which is based the factorisation operation, and we will study termination of the graph series resulting from each of these refinements.

The first, and more general, notion of factorisation introduced in [12] is called *weak-factor graph* (see example on Figure 1).

Definition 4 (V_k^+ and weak-factor graph [12], cf. Figure 1). *Given a k-partite graph $G = (V_0, \ldots, V_{k-1}, E)$ with $k \geq 2$, we define the set V_k^+ as:*

$$V_k^+ = \{\{x_1, \ldots, x_l\} \cup \bigcap_{1 \leq i \leq l} N(x_i) \mid l \geq 2,\ \forall i \in [\![1, l]\!], x_i \in V_{k-1} \text{ and } |\bigcap_{1 \leq i \leq l} N(x_i)| \geq 2\}.$$

The weak-factor graph *G^+ of G is the factorisation of G with respect to V_k^+.*

Unfortunately, it is very easy to find examples of graphs G_0 that generate infinite series for the weak-factor graph operation. This is the reason why [12] introduced two more restricted version of the operator, called *factor graph* and *clean-factor graph*. For the clean-factor operator, they could prove termination of the series for all graphs, but they could not prove it for the factor operator.

Definition 5 (V_k° and factor graph [12]). *Given a k-partite graph $G = (V_0, \ldots, V_{k-1}, E)$ with $k \geq 2$, we define the set V_k° as:*

$$V_k^\circ = \{X \in V_k^+ \text{ such that } |\bigcap_{y \in X \cap V_{k-1}} N_{k-2}(y)| \geq 2\}.$$

The factor graph *G° of G is the factorisation of G with respect to V_k°.*

Definition 6 (V_k^* and clean-factor graph [12]). *Given a k-partite graph $G = (V_0, \ldots, V_{k-1}, E)$ with $k \geq 4$, we define the set V_k^* as:*

$$V_k^* = \{X \in V_k^+ \mid |\bigcap_{y \in X \cap V_{k-1}} N_{k-2}(y)| \geq 2 \text{ and } \forall x, y \in X \cap V_{k-1}, \forall p \in \{0\} \cup [\![2, k-3]\!], N_p(x) = N_p(y$$

The clean-factor graph *G^* of G is the factorisation of G with respect to V_k^* if $k \geq 4$, and $G^* = G^\circ$ if $k \leq 3$.*

This latter definition of the factorisation is much more constrained than the one used in the definition of the weak-factor graph: the conditions to create a new vertex on the higher level are more restrictive. This is the reason why the clean-factor series terminates for all graphs while the weak-factor series does not. But, as mentioned in the introduction, for modelling purposes it is important to find the less constrained definition of the factorisation that guarantees termination for all graphs. This is the reason why [12] asks whether the sole condition $|\bigcap_{y \in X \cap V_{k-1}} N_{k-2}(y)| \geq 2$ required in the definition of the *factor graph* is enough to obtain termination for all graphs. Here we show that it is not and that the iteration of the factor graph operator leads to divergent series in some cases (Section 3). Nevertheless we show that it is possible to significantly weaken the conditions of the clean-factor graph operator and still obtain termination for all graphs (Section 4).

3 The factor series does not always terminate

In this section, we give an example of a graph G for which the factor series does not terminate, thereby answering an open question raised in [12]. The idea of our example is to show by induction that for all integer $k \geq 2$, V_{2k} contains at least 6 elements. To that purpose, for each $k \geq 2$, we prove the existence of 10 particular elements at level V_{2k-1} and 6 particular elements at level V_{2k}. Using these 16 elements, we recursively build 16 new similar elements at levels V_{2k+1} and V_{2k+2}. We do not formally write the induction. Instead, we explicitly build the desired elements of the series of G until the structure of the 16 particular elements is reproduced, which occurs for the first time between levels V_3, V_4 and levels V_5, V_6.

In our inductive construction, we will define some vertices on the upper level V_k as generated by subset of vertices on the lower levels V_l, with $l \leq k-2$ (see Fact 1 below). To that purpose, we need the following definition.

Definition 7 **($Cont_{V_k}(N)$).** *Let $k \geq 1$ and let $N \subseteq \bigcup_{0 \leq i \leq k-1} V_i$. We denote $Cont_{V_k}(N)$ the subset of vertices of V_k whose neighbourhood contains N, i.e. $Cont_{V_k}(N) = \{y \in V_k \mid N \subseteq N(y)\}$.*

Fact 1 (Vertex generated by a subset of vertices) *Let $k \geq 2$ and let $N \subseteq \bigcup_{0 \leq i \leq k-2} V_i$ such that $|N \cap V_{k-2}| \geq 2$. If $|Cont_{V_{k-1}}(N)| \geq 2$, then there exists a (unique) vertex $x \in V_k$ such that $N_{k-1}(x) = Cont_{V_{k-1}}(N)$. This vertex x is called the* vertex of V_k generated by N *and is denoted $x = gen < N >$.*

In our construction, when we define some vertices on the upper level as generated by vertices of the lower levels, we need to check that the generated vertices are distinct. This is the purpose of the following fact.

Fact 2 (Distinguishing property) *Let $k \geq 2$, let $x_1, x_2 \in V_k$ and let $Y \subseteq V_{k-1}$. If $N(x_1) \cap Y \neq N(x_2) \cap Y$, then $x_1 \neq x_2$.*

The statements of Facts 1 and 2 directly follow from the definition of the factor graph and do not need a proof. Let us now start the description of our example and of its factor series.

Level V_0. First, the set V_0 of vertices of G is the set $\{o, o', a_1, ...a_6, b_0, ..., b_6\}$.

Level V_1. Elements of V_1 (i.e. the maximal cliques of G) are:

$v_0 = oo'b_0$
$v_1 = oo'a_2b_1$
$v_2 = oo'a_1a_2b_2$
$v_3 = oo'a_1a_2a_3b_3$
$v_4 = oo'a_1a_2a_3a_4b_4$
$v_5 = oo'a_1a_2a_3a_4a_5b_5$
$v_6 = oo'a_1a_2a_3a_4a_5a_6b_6$

Level V_2. We consider the following set W of the 6 following elements (which is actually the whole set V_2):

$w_1 = v_6v_5$	$oo'a_1a_2a_3a_4a_5$
$w_2 = v_6v_5v_4$	$oo'a_1a_2a_3a_4$
$w_3 = v_6v_5v_4v_3$	$oo'a_1a_2a_3$
$w_4 = v_6v_5v_4v_3v_2$	$oo'a_1a_2$
$w_5 = v_6v_5v_4v_3v_2v_1$	$oo'a_2$
$w_6 = v_6v_5v_4v_3v_2v_1v_0$	oo'

Level V_3. We consider the set X of following elements of V_3:

$x_{1,6} = gen < v_6\ v_5 >$
$x_{1,5} = gen < v_6\ v_5\ a_2 >$
$x_{1,4} = gen < v_6\ v_5\ a_1\ a_2 >$
$x_{1,3} = gen < v_6\ v_5\ a_1\ a_2\ a_3 >$
$x_{1,2} = gen < v_6\ v_5\ a_1\ a_2\ a_3\ a_4 >$

$x_{2,6} = gen < v_6\ v_5\ v_4 >$

$x_{2,5} = gen < v_6\ v_5\ v_4\ a_2 >$
$x_{2,4} = gen < v_6\ v_5\ v_4\ a_1\ a_2 >$
$x_{2,3} = gen < v_6\ v_5\ v_4\ a_1\ a_2\ a_3 >$

$x_{3,4} = gen < v_6\ v_5\ v_4\ v_3\ a_1\ a_2 >$

The intersections with W of the neighbourhoods of each of the ten elements of V_3 defined above are the following:

For $x_{1,6}$: $w_1\ w_2\ w_3\ w_4\ w_5\ w_6$
For $x_{1,5}$: $w_1\ w_2\ w_3\ w_4\ w_5$
For $x_{1,4}$: $w_1\ w_2\ w_3\ w_4$
For $x_{1,3}$: $w_1\ w_2\ w_3$
For $x_{1,2}$: $w_1\ w_2$

For $x_{2,6}$: $w_2\ w_3\ w_4\ w_5\ w_6$
For $x_{2,5}$: $w_2\ w_3\ w_4\ w_5$
For $x_{2,4}$: $w_2\ w_3\ w_4$
For $x_{2,3}$: $w_2\ w_3$

For $x_{3,4}$: $w_3\ w_4$

Since these intersections all contain at least two elements and are pairwise distinct, from Facts 1 and 2, it follows that the ten elements of V_3 defined above are well defined and pairwise distinct.

Level V_4. We consider the set Y of the six elements defined as follows:

$y_1 = gen < Cont_{V_2}(v_6\ v_5\ a_2) >$
$y_2 = gen < Cont_{V_2}(v_6\ v_5\ a_1\ a_2) >$
$y_3 = gen < Cont_{V_2}(v_6\ v_5\ a_1\ a_2\ a_3) >$
$y_4 = gen < Cont_{V_2}(v_6\ v_5\ a_1\ a_2\ a_3\ a_4) >$
$y_5 = gen < Cont_{V_2}(v_6\ v_5\ v_4\ a_1\ a_2\ a_3) >$
$y_6 = gen < Cont_{V_2}(v_6\ v_5\ v_4\ a_1\ a_2) >$

Let us detail as an example the definition of y_1. $Cont_{V_2}(v_6\ v_5\ a_2)$ is the set of elements of V_2 that contains $v_6\ v_5\ a_2$. Element y_1 is defined as the element of V_4 whose neighbourhood at level V_3 is exactly the subset of vertices of V_3 that contain $Cont_{V_2}(v_6\ v_5\ a_2)$. There may be many elements of V_2 containing $v_6\ v_5\ a_2$ but there are at least[4] the elements $w_1\ w_2\ w_3\ w_4\ w_5$ of W, which are the only elements of $Cont_{V_2}(v_6\ v_5\ a_2) \cap W$.

Let us determine the elements of X that are neighbours of y_1, that is the elements of X that contain $Cont_{V_2}(v_6\ v_5\ a_2)$. Clearly, as they are generated by sets of vertices included in $v_6\ v_5\ a_2$, elements $x_{1,6}$ and $x_{1,5}$ of X are neighbours of y_1. On the other hand, from above, elements of V_3 that contain $Cont_{V_2}(v_6\ v_5\ a_2)$ must necessarily contain elements $w_1\ w_2\ w_3\ w_4\ w_5$ of W, which is not the case of the elements of X different from $x_{1,6}$ and $x_{1,5}$ (see construction of level V_3). Therefore, the elements of X that are neighbours of y_1 are exactly $x_{1,6}$ and $x_{1,5}$.

We now determine the set $y_1 \cap W$ of elements of W that are neighbours of y_1. From the definition of factor graph, they are all the elements of W that are contained in all the neighbours of y_1 at level V_3. Since $x_{1,6}$ and $x_{1,5}$ are neighbours of y_1 at level V_3 and since the intersection of their neighbourhoods on W is $w_1\ w_2\ w_3\ w_4\ w_5$ (see construction of level V_3), then $y_1 \cap W$ is included in $w_1\ w_2\ w_3\ w_4\ w_5$. Moreover, by definition, all the neighbours of y_1 at level V_3 contain $Cont_{V_2}(v_6\ v_5\ a_2)$ which itself contains $w_1\ w_2\ w_3\ w_4\ w_5$. As a consequence, $y_1 \cap W$ is exactly the set $w_1\ w_2\ w_3\ w_4\ w_5$.

In the same way, one can check that the intersections with $X \cup W$ of the neighbourhoods of all the six elements of Y defined above are the following:

For y_1:	$x_{1,6}\ x_{1,5}$	$w_1\ w_2\ w_3\ w_4\ w_5$
For y_2:	$x_{1,6}\ x_{1,5}\ x_{1,4}$	$w_1\ w_2\ w_3\ w_4$
For y_3:	$x_{1,6}\ x_{1,5}\ x_{1,4}\ x_{1,3}$	$w_1\ w_2\ w_3$
For y_4:	$x_{1,6}\ x_{1,5}\ x_{1,4}\ x_{1,3}\ x_{1,2}$	$w_1\ w_2$
For y_5:	$x_{1,6}\ x_{1,5}\ x_{1,4}\ x_{1,3}\ x_{2,6}\ x_{2,5}\ x_{2,4}\ x_{2,3}$	$w_2\ w_3$
For y_6:	$x_{1,6}\ x_{1,5}\ x_{1,4}\ x_{2,6}\ x_{2,5}\ x_{2,4}\ x_{3,4}$	$w_3\ w_4$

In particular, one can see that the intersections with X all contain at least two elements and are pairwise distinct. From Facts 1 and 2, it follows that the six elements of V_4 defined above are well defined and pairwise distinct.

The reason why we mentioned the intersections with W is that they

[4] In the special case of the definition of level V_4, it turns out that set W is the whole level V_2, but this is not true in the higher levels. Like for example in the definition of elements of V_6, where $Cont_{V_4}(x_{1,6}\ x_{1,5}\ w_2)$ may not only contain elements of Y but also elements of $V_4 \setminus Y$. Here, we follow the general reasoning that works also for the definition of the higher levels.

are useful for the definitions of the elements of V_5 given below.

Level V_5. We consider the set Z of the following elements:

$z_{1,6} = gen < x_{1,6}\ x_{1,5} >$
$z_{1,5} = gen < x_{1,6}\ x_{1,5}\ w_2 >$
$z_{1,4} = gen < x_{1,6}\ x_{1,5}\ w_1\ w_2 >$
$z_{1,3} = gen < x_{1,6}\ x_{1,5}\ w_1\ w_2\ w_3 >$
$z_{1,2} = gen < x_{1,6}\ x_{1,5}\ w_1\ w_2\ w_3\ w_4 >$

$z_{2,6} = gen < x_{1,6}\ x_{1,5}\ x_{1,4} >$
$z_{2,5} = gen < x_{1,6}\ x_{1,5}\ x_{1,4}\ w_2 >$
$z_{2,4} = gen < x_{1,6}\ x_{1,5}\ x_{1,4}\ w_1\ w_2 >$
$z_{2,3} = gen < x_{1,6}\ x_{1,5}\ x_{1,4}\ w_1\ w_2\ w_3 >$

$z_{3,4} = gen < x_{1,6}\ x_{1,5}\ x_{1,4}\ x_{1,3}\ w_1\ w_2 >$

Consider the intersections of these elements with Y.

For $z_{1,6}$: $y_1\ y_2\ y_3\ y_4\ y_5\ y_6$
For $z_{1,5}$: $y_1\ y_2\ y_3\ y_4\ y_5$
For $z_{1,4}$: $y_1\ y_2\ y_3\ y_4$
For $z_{1,3}$: $y_1\ y_2\ y_3$
For $z_{1,2}$: $y_1\ y_2$

For $z_{2,6}$: $y_2\ y_3\ y_4\ y_5\ y_6$
For $z_{2,5}$: $y_2\ y_3\ y_4\ y_5$
For $z_{2,4}$: $y_2\ y_3\ y_4$
For $z_{2,3}$: $y_2\ y_3$

For $z_{3,4}$: $y_3\ y_4$

As previously, since these intersections all contain at least two elements and are pairwise distinct, from Facts 1 and 2, it follows that the ten elements of V_5 defined above are well defined and pairwise distinct.

Level V_6. We consider the set T of the 6 following elements:

$t_1 = gen < Cont_{V_4}(x_{1,6}\ x_{1,5}\ w_2) >$
$t_2 = gen < Cont_{V_4}(x_{1,6}\ x_{1,5}\ w_1\ w_2) >$
$t_3 = gen < Cont_{V_4}(x_{1,6}\ x_{1,5}\ w_1\ w_2\ w_3) >$
$t_4 = gen < Cont_{V_4}(x_{1,6}\ x_{1,5}\ w_1\ w_2\ w_3\ w_4) >$
$t_5 = gen < Cont_{V_4}(x_{1,6}\ x_{1,5}\ x_{1,4}\ w_1\ w_2\ w_3) >$
$t_6 = gen < Cont_{V_4}(x_{1,6}\ x_{1,5}\ x_{1,4}\ w_1\ w_2) >$.

Using the same reasoning as the one given as example for element y_1 in the construction of level V_4, one can check that the intersections of the 6 elements of T with $Z \cup Y$ are the following:

For t_1:	$z_{1,6}\ z_{1,5}$	$y_1\ y_2\ y_3\ y_4\ y_5$
For t_2:	$z_{1,6}\ z_{1,5}\ z_{1,4}$	$y_1\ y_2\ y_3\ y_4$
For t_3:	$z_{1,6}\ z_{1,5}\ z_{1,4}\ z_{1,3}$	$y_1\ y_2\ y_3$
For t_4:	$z_{1,6}\ z_{1,5}\ z_{1,4}\ z_{1,3}\ z_{1,2}$	$y_1\ y_2$
For t_5:	$z_{1,6}\ z_{1,5}\ z_{1,4}\ z_{1,3}\ z_{2,6}\ z_{2,5}\ z_{2,4}\ z_{2,3}$	$y_2\ y_3$
For t_6:	$z_{1,6}\ z_{1,5}\ z_{1,4}\ z_{2,6}\ z_{2,5}\ z_{2,4}\ z_{3,4}$	$y_3\ y_4$

As before, the intersections with Z all contain at least two elements and are pairwise distinct. From Facts 1 and 2, it follows that the six elements of V_6 defined above are well defined and pairwise distinct. The intersections with Y are useful for the definition of elements of V_7, but we stop our inductive construction here.

The definition and justification of the ten elements on levels V_3 and V_5 are identical, as well as the definition and the justification of the 6 elements on levels V_4 and V_6. Then, by applying this inductive construction process, we can show that the factor series of G never terminates: there are always at least 10 elements on level V_{2k-1} and at least 6 elements on level V_{2k}, for all $k \geq 2$.

4 Termination of the strong-factor series

In the previous section, we showed that the sole cardinality condition, required by the factor graph, on the intersection of neighbourhoods at level V_{k-2} is not sufficient to guarantee termination of the series for all graphs. Nevertheless, in this section, we show that we can get rid of the restrictive equality conditions, required by the clean-factor graph, on the neighbourhoods at level below or equal V_{k-3} and still obtain termination for all graphs. To that purpose, we replace these equality conditions on neighbourhoods by cardinality conditions on their intersections, as for

V_{k-2}. This new factorisation operation for which we prove termination is called *strong-factor graph*.

Definition 8 ($V_k^\bullet$ and strong-factor graph). *Given a k-partite graph $G = (V_0, \ldots, V_{k-1}, E)$ with $k \geq 2$, we define the set $V_k^\bullet$ as:*

$$V_k^\bullet \;=\; \{X \in V_k^+ \text{ such that } \forall l \in [\![0, k-2]\!], |\bigcap_{y \in X \cap V_{k-1}} N_l(y)| \geq 2\}.$$

The strong-factor graph $G^\bullet$ *of G is the factorisation of G with respect to $V_k^\bullet$.*

We now introduce some definitions and notations we need in the rest of the section.

Definition 9 (Intervals of a poset). *For a poset*[5] $(P, \leq)$ *and any two $a, b \in P$, we denote $[a, b] = \{x \in P \mid a \leq x \text{ and } x \leq b\}$ the* interval *defined by a and b.*

Remark 2. Note that in the preceding definition, $[a, b] \neq \varnothing$ iff $a \leq b$.

A family $\mathcal{O}$ of subsets of $V(G)$, namely the non-trivial intersections of maximal cliques of G, will play a key role in the following.

Definition 10 (Non-trivial intersections of maximal cliques). *Let $\mathcal{K}(G)$ be the set of cliques of G. We define the subset $\mathcal{O}$ of $2^{V(G)}$ as follows:*

$$\mathcal{O} = \{O \subseteq V(G) \mid |O| \geq 2 \text{ and } \exists \mathcal{C} \subseteq \mathcal{K}(G), |\mathcal{C}| \geq 2 \text{ and } O = \bigcap_{C \in \mathcal{C}} C\}.$$

We now enhance each level V_k of the strong-factor series with a poset structure. We then show that for every vertex $x \in V_k$, its set of neighbours at level V_{k-1} is an interval of the poset defined on V_{k-1} (Lemma 3 below). This property is at the core of our termination proof.

Definition 11. *Let V_k, $k \geq 2$, be the set of vertices of the $k-th$ part in the multipartite graph of the strong-factor series. We define the order $\preceq_k$ on V_k as: $x \preceq_k x'$ iff $N_{k-1}(x) \subseteq N_{k-1}(x')$.*

Lemmas 1 and 2 below show, for every vertex x at level V_k, $k \geq 3$, the existence of two particular neighbours $y_{min}, y_{max} \in N_{k-1}(x)$ at level V_{k-1}, which are the bounds of the interval of $(V_{k-1}, \preceq_{k-1})$ defined by the neighbours of x at level V_{k-1}.

[5] Partially ordered set, see e.g. [16] for a definition.

Lemma 1. *Let $k \geq 3$ and let $x \in V_k$ in the strong-factor series. Then there exists $y_{min} \in N_{k-1}(x)$ such that $N_{k-2}(y_{min}) = \bigcap_{y \in N_{k-1}(x)} N_{k-2}(y)$.*

Proof. We denote $N_{k-1}(x) = \{y_1, \ldots, y_t\}$ and $\bigcap_{y \in N_{k-1}(x)} N_{k-2}(y) = \{z_1, \ldots, z_s\}$, with $t \geq 2$ and $s \geq 2$ from the definition of the strong-factor graph.

We first prove that there exists an element $y_{min} \in V_{k-1}$ such that $N_{k-2}(y_{min}) = \bigcap_{y \in N_{k-1}(x)} N_{k-2}(y)$, and then we prove that $y_{min} \in N_{k-1}(x)$. For the former part, we aim at proving that $Y_{min} = \{z_1, \ldots, z_s\} \cup \cap_{1 \leq j \leq s} N(z_j)$ is a maximal element of $V^\bullet_{k-1}$.

For any $i \in [\![1, t]\!]$, $N_{k-2}(y_i) \supseteq \{z_1, \ldots, z_s\}$, then for all $l \leq k-3$, $N_l(y_i) \subseteq \cap_{1 \leq j \leq s} N_l(z_j)$. Moreover, since from the definition of the strong-factor graph $|N_l(y_i)| \geq 2$, then for all $l \leq k-3$, $|Y_{min} \cap V_l| \geq 2$. Consequently, $Y_{min} \in V^\bullet_{k-1}$.

Suppose now for contradiction that Y_{min} is not maximal in $V^\bullet_{k-1}$. Let Y'_{min} be the maximal element of $V^\bullet_{k-1}$ containing Y_{min}. Necessarily, $Y'_{min} \cap \bigcup_{0 \leq l \leq k-3} V_l = \cap_{1 \leq j \leq s} N(z_j)$ and there exists $z' \in V_{k-2}$ such that $z' \in Y'_{min} \setminus Y_{min}$ and $N(z') \supseteq \cap_{1 \leq j \leq s} N(z_j)$. Let $i \in [\![1, t]\!]$, since $\{z_1, \ldots, z_s\} \subseteq N_{k-2}(y_i)$ then $\bigcup_{0 \leq l \leq k-3} N_l(y_i) \subseteq \bigcap_{1 \leq j \leq s} N(z_j)$. And since $N(z') \supseteq \cap_{1 \leq j \leq s} N(z_j)$, we have $N(z') \supseteq \bigcup_{0 \leq l \leq k-3} N_l(y_i)$, which implies by maximality of y_i that $z' \in N_{k-2}(y_i)$. Since this holds for any $i \in [\![1, t]\!]$, it follows that $z' \in \bigcap_{y \in N_{k-1}(x)} N_{k-2}(y) = \{z_1, \ldots, z_s\}$, which is a contradiction. Thus, Y_{min} is maximal in $V^\bullet_{k-1}$ and we denote y_{min} the corresponding element of V_{k-1}.

Let us now prove that $y_{min} \in N_{k-1}(x)$. Again, for any $i \in [\![1, t]\!]$, since $\{z_1, \ldots, z_s\} \subseteq N_{k-2}(y_i)$ then $\bigcup_{0 \leq l \leq k-3} N_l(y_i) \subseteq \bigcap_{1 \leq j \leq s} N(z_j)$. Furthermore, since this holds for all $i \in [\![1, t]\!]$, then $\bigcup_{0 \leq l \leq k-3} N_l(x) \subseteq \bigcap_{1 \leq j \leq s} N(z_j) \subseteq N(y_{min})$. Moreover, $N_{k-2}(x) = \bigcap_{y \in N_{k-1}(x)} N_{k-2}(y) = \{z_1, \ldots, z_s\} \subseteq N(y_{min})$. Thus, we have $\bigcup_{0 \leq l \leq k-3} N_l(x) \subseteq N(y_{min})$ and $N_{k-2}(x) \subseteq N(y_{min})$. By maximality of x, it follows that $y_{min} \in N_{k-1}(x)$, which ends the proof.

Lemma 2. *Let $x \in V_k$, with $k \geq 3$, in the strong-factor series. Then there exists $y_{max} \in N_{k-1}(x)$ such that $N_{k-2}(y_{max}) = \bigcup_{y \in N_{k-1}(x)} N_{k-2}(y)$.*

Proof. We denote $N_{k-1}(x) = \{y_1, \ldots, y_t\}$ and $\bigcup_{y \in N_{k-1}(x)} N_{k-2}(y) = \{z_1, \ldots, z_s\}$, with $t \geq 2$ and $s \geq 2$ from the definition of the strong-factor graph.

We first prove that there exists an element $y_{max} \in V_{k-1}$ such that $N_{k-2}(y_{max}) = \bigcup_{y \in N_{k-1}(x)} N_{k-2}(y)$, and then we prove that $y_{max} \in N_{k-1}(x)$. For the former part, we aim at proving that $Y_{max} = \{z_1, \ldots, z_s\} \cup \cap_{1 \leq j \leq s} N(z_j)$ is a maximal element of $V^\bullet_{k-1}$.

Let $l \leq k-3$, by definition, for any $i \in [\![1, t]\!]$, $N_l(x) \subseteq N(y_i)$, and similarly, for any $z \in N_{k-2}(y_i)$, $N_l(y_i) \subseteq N(z)$, and so $N_l(x) \subseteq N(z)$. It follows that $N_l(x) \subseteq \bigcap_{z \in \bigcup_{y \in N_{k-1}(x)} N_{k-2}(y)} N(z) = \cap_{1 \leq j \leq s} N(z_j)$. Then, for all $l \in [\![0, k-3]\!]$, $|Y_{max} \cap V_l| \geq 2$. And consequently $Y_{max} \in V^{\bullet}_{k-1}$.

Suppose now for contradiction that Y_{max} is not maximal in $V^{\bullet}_{k-1}$. Let Y'_{max} be the maximal element of $V^{\bullet}_{k-1}$ containing Y_{max}. Necessarily, $Y'_{max} \cap \bigcup_{0 \leq l \leq k-3} V_l = \cap_{1 \leq j \leq s} N(z_j)$ and there exists $z' \in V_{k-2}$ such that $z' \in Y'_{max} \setminus Y_{max}$. Since we showed above that for all $l \in [\![0, k-3]\!]$, $N_l(x) \subseteq \cap_{1 \leq j \leq s} N(z_j)$, then we have $N(x) \cap \bigcup_{0 \leq l \leq k-3} V_l \subseteq \cap_{1 \leq j \leq s} N(z_j)$ and consequently $N(x) \cap \bigcup_{0 \leq l \leq k-3} V_l \subseteq Y_{max} \subseteq Y'_{max}$. Since we also have $N_{k-2}(x) \subseteq Y_{max} \subseteq Y'_{max}$, then, by maximality of x, the element $y'_{max} \in V_{k-1}$ corresponding to Y'_{max} is such that $y'_{max} \in N_{k-1}(x)$. It follows that $z' \in \bigcup_{y \in N_{k-1}(x)} N_{k-2}(y) = \{z_1, \ldots, z_s\}$ which is a contradiction. Thus, Y_{max} is maximal in $V^{\bullet}_{k-1}$ and we denote y_{max} the corresponding element of V_{k-1}.

Moreover, we have just shown above that the element $y'_{max} \in V_{k-1}$ corresponding to the maximal element Y'_{max} of $V^{\bullet}_{k-1}$ containing Y_{max} belongs to $N_{k-1}(x)$. Since, we also showed that Y_{max} is maximal, we have $y'_{max} = y_{max}$, and then $y_{max} \in N_{k-1}(x)$. This ends the proof of Lemma 2.

Lemmas 1 and 2 allow us to adopt the following notation.

Definition 12. *For any $k \geq 3$ and any vertex $x \in V_k$ in the strong-factor series, we denote $c_{min}(x)$ (resp. $c_{max}(x)$) the unique vertex y_{min} (resp. y_{max}) of $N_{k-1}(x)$ such that $N_{k-2}(y_{min}) = \bigcap_{y \in N_{k-1}(x)} N_{k-2}(y)$ (resp. $N_{k-2}(y_{max}) = \bigcup_{y \in N_{k-1}(x)} N_{k-2}(y)$).*

Remark 3. Note that since there are at least two distinct vertices in $N_{k-1}(x)$, necessarily, $c_{min}(x)$ and $c_{max}(x)$ are distinct and we have $c_{min}(x) \prec_{k-1} c_{max}(x)$.

Based on Lemmas 1 and 2 we are now able, for any $k \geq 3$ and for any vertex $x \in V_k$, to entirely characterise the neighbourhood of x at level V_{k-1}. Lemma 3 below states that it is an interval of the partial order defined on V_{k-1}. This structural property of the multipartite graph generated by the strong-factor series is the keystone of our termination proof (Theorem 1).

Lemma 3. *Let $x \in V_k$, with $k \geq 3$, in the strong-factor series. Then $N_{k-1}(x) = \{y \in V_{k-1} \mid c_{min}(x) \preceq_{k-1} y \preceq_{k-1} c_{max}(x)\}$.*

Proof. Remember that by definition, $c_{min}(x) \preceq_{k-1} y \preceq_{k-1} c_{max}(x)$ is equivalent to $N_{k-2}(c_{min}(x)) \subseteq N_{k-2}(y) \subseteq N_{k-2}(c_{max}(x))$. Clearly, from the definitions of $c_{min}(x)$ and $c_{max}(x)$ (see Lemmas 1 and 2), we have $N_{k-1}(x) \subseteq \{y \in V_{k-1} \mid c_{min}(x) \preceq_{k-1} y \preceq_{k-1} c_{max}(x)\}$.

Conversely, let $y \in V_{k-1}$ such that $N_{k-2}(c_{min}(x)) \subseteq N_{k-2}(y) \subseteq N_{k-2}(c_{max}(x))$. Since $N_{k-2}(y) \subseteq N_{k-2}(c_{max}(x))$, then $\bigcup_{0\leq l\leq k-3} N_l(y) \supseteq \bigcup_{0\leq l\leq k-3} N_l(c_{max}(x))$. And since $c_{max}(x) \in N_{k-1}(x)$, then $\bigcup_{0\leq l\leq k-3} N_l(c_{max}(x)) \supseteq \bigcup_{0\leq l\leq k-3} N_l(x)$. Thus, we have $\bigcup_{0\leq l\leq k-3} N_l(y) \supseteq \bigcup_{0\leq l\leq k-3} N_l(x)$ and since by definition of y we also have $N_{k-2}(y) \supseteq N_{k-2}(c_{min}(x)) = N_{k-2}(x)$, then necessarily, by maximality of x, $y \in N_{k-1}(x)$.

For our proof of termination, we will need the following property of nested families of intervals of a partial order.

Lemma 4. *Let $(P, \leq)$ be a partially ordered set. Let $\{[a_i, b_i]\}_{1\leq i\leq p}$ a family of p distinct intervals of P not reduced to a singleton and totally ordered for inclusion, that is $a_p \leq a_{p-1} \leq \ldots \leq a_2 \leq a_1 < b_1 \leq b_2 \leq \ldots \leq b_{p-1} \leq b_p$. Then, there exist $p+1$ elements in set $\{a_i \mid 1 \leq i \leq p\} \cup \{b_i \mid 1 \leq i \leq p\}$ that are pairwise distinct and totally ordered for $\leq$.*

Proof. The condition $a_p \leq a_{p-1} \leq \ldots \leq a_2 \leq a_1 < b_1 \leq b_2 \leq \ldots \leq b_{p-1} \leq b_p$ involves $2p-1$ inequalities, one of which is strict. Suppose for contradiction that among the remaining $2(p-1)$ large inequalities, at least p of them are actually equalities. Then, from the pigeon-hole principle, there necessarily exists an index $i \in [\![1, p-1]\!]$ such that $a_{i+1} = a_i$ and $b_i = b_{i+1}$. Then, the two intervals $[a_i, b_i]$ and $[a_{i+1}, b_{i+1}]$ are not distinct, which is a contradiction. Thus, there are at most $p-1$ large inequalities that are actually equalities, and it follows that there are at least $p-1$ large inequalities that are actually strict inequalities. Together with the central strict inequality, it gives a set of p strict inequalities that define a family of $p+1$ distinct and totally ordered elements in $\{a_i \mid 1 \leq i \leq p\} \cup \{b_i \mid 1 \leq i \leq p\}$.

Using Lemmas 3 and 4 we can now achieve the proof of termination of the strong-factor series.

Theorem 1. *The strong-factor series terminates for all graphs.*

Proof. We will prove that, for any initial graph G, there exists an integer k such that $V_k = \varnothing$. To that purpose, for any $k \geq 3$ such that $V_k \neq \varnothing$, we prove by induction that the following statement $H(l) =$ "V_{k-l} contains at least $l+1$ distinct elements totally ordered for $\preceq_{k-l}$" holds for all $l \in [\![1, k-2]\!]$. For $l = 1$, since there exists a vertex $x \in V_k$, from Remark 3, $c_{min}(x)$ and $c_{max}(x)$ are two distinct vertices of V_{k-1} that are comparable for $\preceq_{k-1}$.

Let us now assume that the statement $H(l)$ holds for some $l \geq 1$ and $l \leq k-3$ and let us show that it holds for $l+1$ as well. Let us denote $x_1 \preceq_{k-l} \dots \preceq_{k-l} x_{l+1}$ the $l+1$ distinct elements of V_{k-l} totally ordered for $\preceq_{k-l}$. From Lemma 3, for any $i \in [\![1, l+1]\!]$, $N_{k-l-1}(x_i) = [a_i, b_i]$, where $a_i = c_{min}(x_i) \in V_{k-l-1}$ and $b_i = c_{max}(x_i) \in V_{k-l-1}$, and where the interval is to be taken in the sense of order $\preceq_{k-l-1}$ on V_{k-l-1}. From Remark 3, these intervals are not reduced to singletons and they are all distinct, as the x_i's are. Moreover, since $x_1 \preceq_{k-l} \dots \preceq_{k-l} x_{l+1}$, the intervals $[a_i, b_i]$ are totally ordered. Consequently, from Lemma 4, there exist $l+2$ distinct vertices of V_{k-l-1} that are comparable for $\preceq_{k-l-1}$, which ends the induction and shows that $H(l)$ holds for all $l \in [\![1, k-2]\!]$.

Thus, we showed that if $V_k \neq \varnothing$ then, for $l = k-2$, V_2 necessarily contains at least $k-1$ distinct elements totally ordered for $\preceq_2$. On the other hand, [12] showed that $(V_2, \preceq_2)$ is isomorphic to order $\mathcal{L}$, which is the inclusion order on the non-trivial intersections of maximal cliques of graph G (see Section 2). This implies that if $V_k \neq \varnothing$ then the height h of order $\mathcal{L}$ is at least $k-1$. It follows that for all $k > h+1$, we have $V_k = \varnothing$, which ends the proof of the theorem.

It is important to note that the proof of termination above also gives an upper bound on the time of termination of the series.

Corollary 1. *Let G be a graph, let $\mathcal{L}$ be the inclusion order of the non-trivial intersections of its maximal cliques and let h be the height of $\mathcal{L}$. Then, level V_{h+2} of the strong-factor series is always empty.*

This bound of $h+1$ given by Corollary 1 for the index of the last non-empty level of the strong-factor series of a graph G has to be compared with the bound obtained in [12] for the clean-factor series. It turns out that Theorem 2 of [12] implies that there exists at least one element on level V_{h+1} in the clean-factor series of G, which gives the following comparison between the times of termination of the two series.

Corollary 2. *The strong-factor series never terminates later than the clean-factor series. In other words, if $V_k = \varnothing$ in the clean-factor series for some $k \geq 2$ then $V_k = \varnothing$ in the strong-factor series as well.*

In addition to the results on termination of the strong-factor series obtained above, we are able to give a complete characterisation of the levels V_k of the strong-factor series that is worth of interest in itself (Theorem 2 below). We already showed that the neighbours at level V_{k-1} of a vertex $x \in V_k$ are an interval of order $(V_{k-1}, \preceq_{k-1})$ (Lemma 3). Theorem 2 establishes that the converse is also true: all non-trivial intervals of $(V_{k-1}, \preceq_{k-1})$ define an element at level V_k.

Theorem 2. *For any $k \geq 3$, the function $\phi : x \mapsto N_{k-1}(x)$ is an order isomorphism*[6] *from $(V_k, \preceq_k)$ to $(\mathcal{I}_{k-1}, \subseteq)$, where $\mathcal{I}_{k-1}$ is the set of non-trivial intervals (i.e. having at least two elements) of $(V_{k-1}, \preceq_{k-1})$.*

Proof. Let $x \in V_k$, with $k \geq 3$. Clearly, from Lemma 3 and Remark 3, $\phi(x)$ is a non-trivial interval of $(V_{k-1}, \preceq_{k-1})$, that is $\phi(x) \in \mathcal{I}_{k-1}$. Moreover, function ϕ is injective as, from the definition of the strong-factor graph, two vertices of V_k having the same neighbourhood on V_{k-1} are necessarily equal. Finally, note that since order $\preceq_k$ is precisely the inclusion order on the neighbourhoods at level V_{k-1} (see Definition 11), then ϕ is automatically an order morphism.

The only thing that remains to be shown in order to prove Theorem 2 is that ϕ is surjective from V_k onto $\mathcal{I}_{k-1}$. Let $[y_a, y_b] \in \mathcal{I}_{k-1}$, that is $y_a, y_b \in V_{k-1}$ and $y_a \prec_{k-1} y_b$. We denote $\{y_1, \ldots, y_s\} = \{y \in V_{k-1} \mid y_a \preceq_{k-1} y \preceq_{k-1} y_b\}$. We will prove that $X = \{y_1, \ldots, y_s\} \cup \cap_{1 \leq j \leq s} N(y_j)$ is a maximal element of $V_k^\bullet$.

First, let us prove that X is an element of $V_k^\bullet$. Since y_a and y_b are distinct, then $|X \cap V_{k-1}| \geq 2$. Since $y_a \preceq_{k-1} y$ we have $N_{k-2}(y_a) \subseteq N_{k-2}(y)$ for all $y \in \{y_1, \ldots, y_s\}$, which implies that $X \cap V_{k-2} \supseteq N_{k-2}(y_a)$. As $y_a \in V_{k-1}$, we have $|N_{k-2}(y_a)| \geq 2$, and then $|X \cap V_{k-2}| \geq 2$. Moreover, since for all $y \in \{y_1, \ldots, y_s\}$, $y \preceq_{k-1} y_b$ then $N_{k-2}(y) \subseteq N_{k-2}(y_b)$, and from the definition of the strong-factor graph, $N_l(y) \supseteq N_l(y_b)$, for all $l \leq k-3$. Since this holds for all $y \in \{y_1, \ldots, y_s\}$, it follows that $X \cap V_l \supseteq N_l(y_b)$, for all $l \leq k-3$. Finally, as $y_b \in V_{k-1}$, we have $|N_l(y_b)| \geq 2$, and so $|X \cap V_l| \geq 2$, for all $l \leq k-3$. Thus, $X \in V_k^\bullet$.

We now prove that X is maximal in $V_k^\bullet$. Let X' be the maximal element of $V_k^\bullet$ containing X and let $z \in X' \cap V_{k-1}$. Necessarily, $N_{k-2}(z) \supseteq X \cap V_{k-2} = N_{k-2}(y_a)$, from the definition of X. On the other hand, since for all $y \in X \cap V_{k-1}$, $N_{k-2}(y) \subseteq N_{k-2}(y_b)$, it follows that $N_l(y) \supseteq N_l(y_b)$, for all $l \leq k-3$. Furthermore, since this holds for all $y \in X \cap V_{k-1}$, and also in particular for y_b, we have $X \cap V_l = N_l(y_b)$, for all $l \leq k-3$. As a consequence, since X' contains X and $z \in X' \cap V_{k-1}$, we have that

[6] Let us recall that a poset $(P, \leq_P)$ is order-isomorphic to another poset $(Q, \leq_Q)$ if there exists a bijection f from P to Q which preserves the orders on the two posets, that is $x \leq_P y$ iff $f(x) \leq_Q f(y)$.

$N_l(z) \supseteq X \cap V_l = N_l(y_b)$, for all $l \leq k-3$. Then, necessarily, for all $z_c \in N_{k-2}(z)$, $N_l(z_c) \supseteq N_l(y_b)$, for all $l \leq k-3$. Then, by maximality of y_b, any $z_c \in N_{k-2}(z)$ belongs to $N_{k-2}(y_b)$, that is $N_{k-2}(z) \subseteq N_{k-2}(y_b)$. Since we already showed that $N_{k-2}(y_a) \subseteq N_{k-2}(z)$, it follows that $y_a \preceq_{k-1} z \preceq_{k-1} y_b$, that is $z \in X \cap V_{k-1}$. Thus, $X = X'$ and X is maximal in $V_k^\bullet$. And since, by definition, the element $x \in V_k$ corresponding to X is such that $\phi(x) = [y_a, y_b]$, this ends the proof that ϕ is surjective, as well as the proof of Theorem 2.

Theorem 2 is interesting for two main reasons. Firstly, it gives a characterisation of the strong-factor series which is simpler than its original definition in terms of maximal bicliques in multipartite graphs. Secondly, this characterisation provides an efficient way to compute the strong-factor series: one does not need to go through the computation of the maximal bicliques of the multipartite graphs of the series (which is a NP-complete problem in general) but only to compute the non-trivial intervals of the orders $(V_k, \preceq_k)$, which is feasible in (low) polynomial time.

5 Conclusion and perspectives

In this paper, we studied the possibility to force the termination of the weak-factor series by adding some additional constraints, as light as possible, to the definition of the operator. In [12], the authors had already shown that it is possible to force termination by requiring equality of the neighbourhoods, at all levels of the series, of the vertices involved in the factorisation step defining the operator. Here, we showed that it is possible to strongly relax these constraints and still guarantee termination of the series for all graphs, and within the same termination time.

More explicitly, in the strong-factor operator introduced here, we replaced the equality constraints on the neighbourhoods of the vertices of V_{k-1} involved in the factorisation step by *cardinality constraints* only requiring that these vertices share at least two neighbours in common at all previous levels V_l of the series, with $l \leq k-2$. Moreover, in section 3, we showed that requiring the cardinality constraint only at level V_{k-2} leads some series to be infinite. Consequently, in the perspective of determining the minimum constraints that guarantee termination of the series for all graphs, one of the main questions raised by our work is to know whether there exists a constant $c \geq 3$ such that requiring the cardinality constraint on levels V_l with $k-c \leq l \leq k-2$ force termination for all graphs.

References

1. Claudson F. Bornstein and Jayme L. Szwarcfiter. On clique convergent graphs. *Graphs and Combinatorics*, 11:213–220, 1995.
2. Mrcia R. Cerioli. Clique graphs and edge-clique graphs. *Electronic Notes in Discrete Mathematics*, 13:34 – 37, 2003. 2nd Cologne-Twente Workshop on Graphs and Combinatorial Optimization.
3. Gary Chartrand, S. F. Kapoor, Terry A. McKee, and Farrokh Saba. Edge-clique graphs. *Graphs and Combinatorics*, 7:253–264, 1991.
4. Christophe Crespelle, Matthieu Latapy, and Thi Ha Duong Phan. On the termination of some biclique operators on multipartite graphs. *Discrete Applied Mathematics*, 2015. To appear.
5. Marina Groshaus and Leandro P. Montero. The number of convergent graphs under the biclique operator with no twin vertices is finite. *Electronic Notes in Discrete Mathematics*, 35:241–246, 2009.
6. Marina Groshaus and Leandro P. Montero. On the iterated biclique operator. *Journal of Graph Theory*, 73(2):181–190, 2013.
7. Jean-Loup Guillaume and Matthieu Latapy. Bipartite structure of *all* complex networks. *Information Processing Letters (IPL)*, 90(5):215–221, 2004.
8. Jean-Loup Guillaume and Matthieu Latapy. Bipartite graphs as models of complex networks. *Physica A*, 371:795–813, 2006.
9. F. Larrión, V. Neumann-Lara, and M. A. Pizaña. Clique divergent clockwork graphs and partial orders. *Discrete Appl. Math.*, 141:195–207, May 2004.
10. Francisco Larrión, Célia Picinin de Mello, Aurora Morgana, Victor Neumann-Lara, and Miguel A. Pizaña. The clique operator on cographs and serial graphs. *Discrete Mathematics*, 282(1-3):183–191, 2004.
11. Francisco Larrión, Miguel A. Pizaña, and R. Villarroel-Flores. Contractibility and the clique graph operator. *Discrete Mathematics*, 308(16):3461–3469, 2008.
12. Matthieu Latapy, Thi Ha Duong Phan, Christophe Crespelle, and Thanh Qui Nguyen. Termination of multipartite graph series arising from complex network modelling. In *COCOA'10*, volume 6508 (Part I) of *Lecture Notes in Computer Science*, pages 1–10, 2010.
13. Min Chih Lin, Francisco J. Soulignac, and Jayme L. Szwarcfiter. The clique operator on circular-arc graphs. *Discrete Appl. Math.*, 158:1259–1267, June 2010.
14. Erich Prisner. Convergence of iterated clique graphs. *Discrete Mathematics*, 103(2):199 – 207, 1992.
15. J. L. Szwarcfiter. A survey on clique graphs. In Jonathan M. Borwein, Peter Borwein, Bruce A. Reed, and Cludia L. Sales, editors, *Recent Advances in Algorithms and Combinatorics*, CMS Books in Mathematics, pages 109–136. Springer New York, 2003.
16. William T. Trotter. *Combinatorics and Partially Ordered Sets: Dimension Theory.* J. Hopkins University Press, 1992.