Abstract Domains in Constraint Programming

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Abstract

Constraint Programming aims at solving hard combinatorial problems, with a computation time increasing in practice exponentially. The methods are today efficient enough to solve large industrial problems, in a generic framework. However, solvers are dedicated to a single variable type: integer or real. Solving mixed problems relies on *ad hoc* transformations. In another field, Abstract Interpretation offers tools to prove program properties, by studying an abstraction of their concrete semantics, that is, the set of possible values of the variables during an execution. Various representations for these abstractions have been proposed. They are called abstract domains. Abstract domains can mix any type of variables, and even represent relations between the variables. In this book, we define abstract domains for Constraint Programming, so as to build a generic solving method, dealing with both integer and real variables. We can also study the octagons abstract domain, already defined in Abstract Interpretation. Guiding the search by the octagonal relations, we obtain good results on a continuous benchmark. In a second part, we define our solving method using Abstract Interpretation techniques, in order to include existing abstract domains. Our solver, AbSolute, is able to solve mixed problems and use relational domains.

Chapter 1

Introduction

Recent advances in computer science are undeniable. Some are visible, others less known to the general public: today, we are able to quickly solve many problems known to be difficult (requiring a long computation time). For instance, it is possible to automatically place in a tens of seconds, thousands of objects of various shapes in a minimum number of containers, and respecting specific constraints: accessibility of goods, non-crush, *etc.* [BEL 07]. Constraint Programming formalizes such problems using constraints that describe a result one want to achieve (accessibility of certain objects, for example). These constraints come with efficient algorithms to solve greatly combinatorial problems. In another research area, semantics, Abstract Interpretation (AI) attacks an insoluble problem in the general case: the correction of programs. With strong theoretical tools developed from its creation (fixpoint theorems), Abstract Interpretation manages to prove properties of programs. In this area, too, the effectiveness of methods allows impressive applications to be solved: tools in Abstract Interpretation have, for instance managed to prove that there was no overflow errors in the flight controls of the Airbus A380 which contain almost 500 000 lines of code.

The work presented in this book is at the interface between Constraint Programming and Abstract Interpretation, two research areas in computer science with a priori quite different problematic. In Constraint Programming, the goal is usually to get good computation time for problems that are in general NP, or extend existing tools to handle more problems. In Abstract Interpretation, the goal is to analyze very large programs by capturing a maximum of properties. Yet there is in these two disciplines a common concern: identifying an impossible or difficult (computationally) space to exactly compute: the solutions set in Constraint Programming, the semantics of the program in Abstract Interpretation. It is about computing the relevant over-approximations of this space. Constraint Programming proposes methods to carefully surround this space (consistency and propagation), but always with Cartesian over-approximations (boxes in $\mathbb{R}^n$ or $\mathbb{Z}^n$). Abstract Interpretation uses often less accurate over-approximations but not only Cartesian: they may have various different shapes (boxes but also octagons, ellipsoids, *etc.*). These non-Cartesian approximation allows more properties to be captured.

In this book, we exploit the similarities of these over-approximation methods to integrate Abstract Interpretation tools in methods of Constraint Programming. We redefine tools in Constraint Programming from notions of Abstract Interpretation (abstract domains). This is not only an intellectual exercise. Indeed, by generalizing the description of over-approximations, there is a significant gain in expressiveness in Constraint Programming. In particular, as the problems are treated uniformly for real and integer variables, which is not currently the case. We also develop the octagon abstract domain, showing that it is possible to exploit the relationships captured by this particular domain to solve continuous problems more effectively. Finally, we perform the opposite task: we define Constraint Programming as an abstract operation in Abstract Interpretation, and develop a solver capable of handling practically all abstract domains.

1.1. Context

As mentioned earlier, the Constraint Programming and Abstract Interpretation have a common concern: compute efficiently and the most accurately possible an approximation of a difficult or impossible space. However, the issues and problems of these two areas are different, as well as their fields of application.

1.1.1. *Constraint Programming*

Constraint Programming, whose origins date back to 1974 [MON 74], is based on the formalization of problems such as a combination of first-order logic formulas, the constraints. A constraint defines a relationship between the variables of a problem: for example, two objects placed in the same container have an empty geometric intersection, or heavy objects should be placed under a fragile object. So this is a declarative programming. Constraint Programming provides efficient generic solution methods for many combinatorial problems. Academic and industrial applications are varied: job shop scheduling problems [GRI 11, HER 11a], design of substitution tables in cryptography [RAM 11], scheduling problems [STØ 11], prediction of the RNA secondary structure in biology [PER 09], optical network design [PEL 09], or automatic harmonization in music [PAC 01].

One of the limitations of the expressiveness of Constraint Programming methods is that they are dedicated to the nature of the problem: solvers used for discrete variables problems are fundamentally different from techniques dedicated to continuous variables problems. In a way, the semantics of the problem is different depending on whether one deals with discrete or continuous problems.

However, many industrial problems are mixed: they contain both integer and real variables. This is the case for example of the problem of fast power grid repair after a natural disasters [SIM 12] to restore the power as quickly as possible in the affected areas. In this problem, we try to establish a plan of action and determine the routes that should be used by repair crews. Some of the variables are discrete, for example, to each device (generator, line) is associated to a Boolean variable indicating whether it is operational or not. Others are real, as the electrical power on a line. Another example of application is the design of the topology of a multicast transmission network [CHI 08]: we want to design a network that is reliable. A network is said reliable, when it is still effective when one of its components is defective, so that all user communications can pass into the network with the least possible delay. Again, some of the variables are integers, the number of lines in the network, others are continuous, the flow of information passing over the network average.

The convergence of discrete and continuous constraints in Constraint Programming is both an industrial need and a scientific challenge.

1.1.2. *Abstract Interpretation*

The basis of Abstract Interpretation were established in 1976 by Cousot and Cousot [COU 76]. Abstract Interpretation is the theory of semantic approximation [COU 77b] which one of the applications is programs proof. The goal is to verify and prove that a program does not contain a bugs, that is to say, runtime errors. Industrial stakes are high. Indeed, many bugs have made history as the bug of 2000, due to system design error. On January 1st 2000, some systems showed the date of January 1st 1900. This bug should be repeated on January 19th 2038 on some UNIX systems [ROB 99]. Another example of a bug is that of the infamous inaugural flight of the Ariane 5 rocket, which, due to an error in the navigation system, caused the destruction of the rocket only 40 seconds after takeoff.

Every day new softwares are developed, corresponding to thousands or millions of lines of code. To test or verify these programs by hand would require considerable time. The soundness of programs can not be proven in a generic way, thus Abstract Interpretation implements methods to automatically analyze certain properties of a program. The analyzers are based on operations on the semantics of programs, that is, the set of values that can be taken by the variables of the program during its execution. By computing an over-approximations of these semantics, the analyzer can for example prove that the variables do not take values beyond the permitted ranges (*overflow*).

Many analyzers are developed and used for various application areas such as aerospace [LAC 98, SOU 07], radiation [POL 06] or particle physics [COV 11].

1.2. Problematic

In this book, we focus on Constraint Programming solving methods called complete, that find the solution set or prove that it is empty, if necessary. These methods are based on an exhaustive search of the space of all possible values, also called search space. Using operations to restrict the space to visit (consistency and propagation), these methods can be accelerated. Existing methods are dedicated to a certain type of variable, discrete or continuous. Facing a mixed problem, containing both discrete and continuous variables, Constraint Programming offers no real solution and techniques available are often limited. Typically, variables are artificially transformed so that they are all discrete as in the solver Choco [TEA 10], or all continuous as in the solver RealPaver [GRA 06]. In Abstract Interpretation analyzed programs often contain, if not always, different types of variables. Theories of Abstract Interpretation integrate many types of domains, and helped develop analyzers uniformly dealing with discrete and continuous variables.

We propose to draw inspiration from the work of the Abstract Interpretation community on the different types of domains to provide new solving methods in Constraint Programming. These new methods should be able, in particular, to approximate with various shapes and solve mixed problems.

1.3. Outline of this book

This book is organized as follows. Chapter 2 gives the mandatory notions of Abstract Interpretation and Constraint Programming to understand our work and an analysis of the similarities and differences between these two research areas. Based on the similarities identified between Constraint Programming and Abstract Interpretation, we define abstract domains for Constraint Programming, with a resolution based on these abstract domains, Chapter 3. The use in Constraint Programming of an example of abstract domain existing Abstract Interpretation, the octagons, is detailed Chapter 4. Chapter 5 gives the solving method implementation details presented chapter 3 for octagons. Finally, Chapter 6 redefines the concepts of Constraint Programming using the techniques and tools available in Abstract Interpretation, to define a method called abstract resolution. A prototype implementation is finally presented as well as experimental results.

1.4. Contributions

The work of this book aims at designing new solving techniques for Constraint Programming. There are two parts in this work. In the first part, the abstract domains of Abstract Interpretation are defined for Constraint Programming, so as different operators required for the solving process. These new definitions allow us to define a uniform resolution framework that no longer depend on the variables type or on the representation of the variables values. An example of a solver using the octagon abstract domain and respecting the framework is implemented in a continuous solver Ibex [CHA 09a], and tested on examples of continuous problems. In the second part, the different Constraint Programming operators needed to solve are defined in Abstract Interpretation, allowing us to define a solving method with the existing operators in Abstract Interpretation. This method was then implemented over Apron [JEA 09], a library of abstract domains.

Most theoretical and practical results of the chapters 3, 4, 5 and 6 are the subject of publications in conferences or journals [TRU 10, PEL 11, PEL 13, PEL 14].

Chapter 2

State of the Art

In this chapter we present the notions upon which Abstract Interpretation is based and the principles of Constraint Programming. We do not do an exhaustive presentation of both areas, but rather give the notions needed for the understanding of this book. The concepts discussed include those of partially ordered set, lattice and fixpoint, which are at the basis of the underlying theories in both fields. It also includes the in place tools, such as narrowing and widening operators in Abstract Interpretation or consistency and splitting operator in Constraint Programming. Finally, this chapter presents a analysis of the similitudes between Abstract Interpretation and Constraint Programming upon which rely the works presented in this book.

2.1. Abstract Interpretation

The founding principles of Abstract Interpretation were introduced in 1976 by Patrick and Radhia Cousot [COU 76]. In this section, we only present some aspects of Abstract Interpretation that will be needed afterward. For a more complete presentation see [COU 92a, COU 77a].

2.1.1. *Introduction to Abstract Interpretation*

One of Abstract Interpretation (AI) applications is to automatically prove that a certain type of bugs does not exist in a program, that there is no error during a program execution. Let's see on an example.

Example 2.1.1 – Consider the following program:

```
1: real $x, y$
```

```
2: $x \leftarrow 2$
3: $y \leftarrow 5$
4: $y \leftarrow y * (x - 2)$
5: $y \leftarrow x/y$
```

The backtrace for this program is:

line	x	y	
1	?	?	
2	2	?	
3	2	5	
4	2	0	
5	2	NaN	Error: division by zero

On toy examples like this one, the backtrace allows one to quickly detect that the program contains errors. However, real life programs are more complex and larger in term of lines of code, it is thus impossible to try all the possible executions. Moreover, the Halting theorem states that it is undecidable to prove that a program terminates.

Nowadays, computer science is omnipresent and critical programs may contain thousands or even millions of lines of code [HAV 09]. In these programs, execution errors are directly translate into significant cost. For example, in 1996, the destruction of the Ariane 5 rocket was due to an integer overflow [Ari96]. Or in 1991, American Patriot missiles failed to destroy an enemy Scud killing 28 soldiers due to a rounding error that has been propagated through the computations [Mis92]. We must therefore ensure that such programs do not have any executions errors. Moreover, this shall be done in a reasonable time without having to run the program. Indeed, sending probes in space just to check if the program is correct, in the sense that it does not contain execution errors, is not a viable solution form an economical and ecological point of view. This is where Abstract Interpretation comes into play. One of its applications is to verify that a program is correct during the compilation process and thus before it is executed. The main idea is to study the values that can be taken by the variables throughout the program. We call *semantics* the set of these values and *specification* the set of all the desired behaviors such as never divided by zero. If the semantics meets all the given specifications then we can say that the program is correct.

An important application of Abstract Interpretation is the design of static programs analyzers that are correct and complete. An analyzer is said to be correct if it answers that a program is correct only when this program does not contain any execution error. There exist several static analyzers, we distinguish two types, the correct ones such as Astrée [BER 10] and Polyspace [Pol10], and the non-correct ones such as Coverity [COV 03]. These analyzers all have industrials applications. For instance, Astrée was able to automatically prove the absence of runtime errors in the primary flight control software of the Airbus A340 fly-by-wire system. And more recently, it analyzed the electric flight control code for the Airbus A380 [ALB 05, SOU 07]. Polyspace was used to analyze the flight control code for the Ariane 502 rocket [LAC 98] and verify security softwares of nuclear installations [POL 06]. As for Coverity, it has been used to verify the code of the curiosity Mars Rover [COV 12] and ensure the accuracy of the Large Hadron Collider (LHC) software [COV 11], the particle accelerator that led to the discovery of the Higgs Boson particle.

Computing the real semantics, called *concrete semantics*, is very costly and undecidable in the general case. Indeed, Rice's theorem states that any non-trivial property formulated only on the inputs and outputs of a program is undecidable. Thus, the analyzers compute an over-approximation of the concrete semantics, the *abstract semantics*. The first step is to associate to each instruction of the program a function. This function modifies the set of possible values for the variables with respect to the instruction. The program thus becomes a composition of these functions and the set of observable behaviors corresponds to a fixpoint of this composition. The second step is to define an abstract domain to restrict the expressivity by keeping only a subset of the properties on the program variables. An abstract domain is a computable data structure used to depict some of the program properties. Moreover, abstract domains come with efficient algorithms to compute different operations on the concrete semantics and allowing the fixpoint to be computed in a finite time. The analyzer always observe a superset of the program behaviors, thus all the found properties by the analyzer are verified for the program, however it can omit some of them.

Example 2.1.2 – Consider the following program:

```
1: real $x, y$
2: $x \leftarrow random(1, 10)$
3: **if** $x$ mod $2 = 0$ **then**
4:     $y \leftarrow x/2$
5: **else**
6:     $y \leftarrow x - 1$
7: **end if**
```

As the variable x takes its value between 1 and 10 (instruction 2), one can deduce that the variable y takes its value between 0 and 8. However, the fact that for every execution $y < x$ is an omitted property by the intervals abstract domain but can be found with the polyhedra abstract domain.

We present in the following subsections the theoretical notions upon which the Abstract Interpretation relies.

2.1.2. *General Presentation*

The Abstract Interpretation underlying theory used notions of fixpoint and lattices. These notions are re-called in this section.

2.1.2.1. *Lattices*

Lattices are a well-known notion in Computer science. Here they are used to express operations on abstract domains and require some properties, such as being closed and complete.

Definition 2.1.1 (Poset)**.** A relation $\sqsubseteq$ on a non-empty set $\mathcal{D}$ is a *partial order* (po) if and only if it is reflexive, antisymmetric and transitive. A set with a partial order is called *partially ordered set* (poset). If they exist, we denote by $\perp$ the least element and by $\top$ the greatest element of $\mathcal{D}$.

Example 2.1.3 – Let $\mathbb{F}$ be the set of floating-point numbers according to the IEEE norm [GOL 91]. For $a, b \in \mathbb{F}$, we can define $[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}$ the real interval delimited by the floating-points numbers a and b, and $\mathbb{I} = \{[a, b], a, b \in \mathbb{F}\}$ the set of intervals. Given an interval $I \in \mathbb{I}$, we denote by $\underline{I}$ (resp. $\overline{I}$) its lower (resp. upper) bound and for all point x, $\underline{x}$ (resp. $\overline{x}$) its lower (resp. upper) floating approximation.

Let $\mathbb{I}^n$ be the set of Cartesian product of n intervals. The set $\mathbb{I}^n$ with the inclusion relation $\subseteq$ is a partially ordered set.

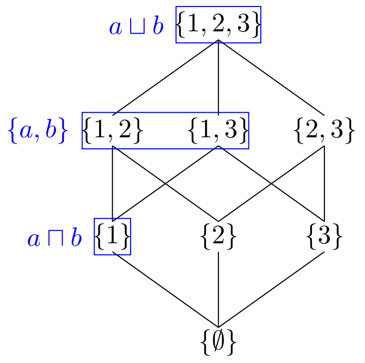
Remark 2.1.1 – Note that the inclusion relation $\subseteq$ is a partial order. Thus, for any non-empty set E, $\mathcal{P}(E)$ with this relation is a partially ordered set.

Definition 2.1.2 (Lattice)**.** A partially ordered set $(\mathcal{D}, \sqsubseteq, \sqcup, \sqcap)$ is a *lattice* if and only if for $a, b \in \mathcal{D}$, the pair $\{a, b\}$ has a least upper bound (*lub*) denoted by $a \sqcup b$, and a greatest lower bound (*glb*) denoted by $a \sqcap b$. A lattice is said *complete* if and only if any subset has both a least upper bound and a greatest lower bound.

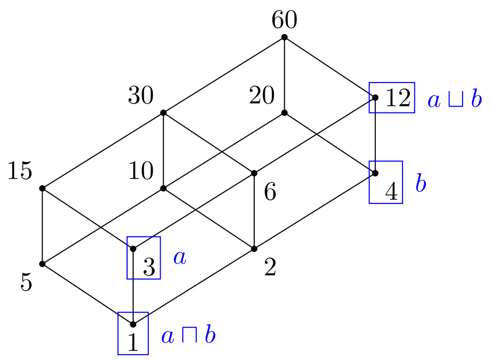
In a lattice, any *finite* subset has a least upper bound and a greatest lower bound. In a complete lattice, any subset has a least upper bound and a greatest lower bound, even if the subset is not finite. A complete lattice thus has a greater element denoted by $\top$, and a least element denoted by $\perp$.

Remark 2.1.2 – Notice that any finite lattice is automatically complete.

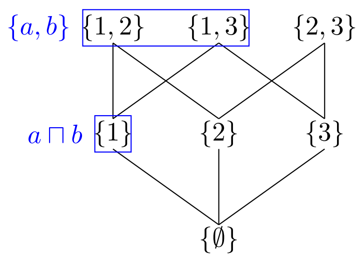
Figure 2.1 gives examples of partially ordered sets represented with Hasse diagrams. The first one (Figure 2.1(a)) corresponds to the power sets of the set $\{1, 2, 3\}$ with the set inclusion $\subseteq$. This partially ordered set is finite and has a least element $\{\emptyset\}$



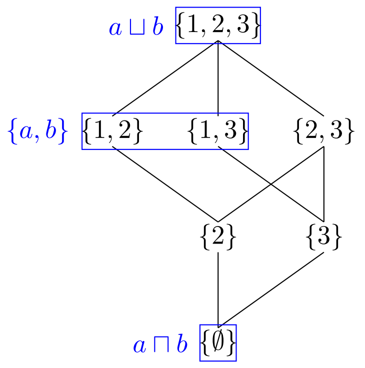
(a) $(\mathcal{P}(\{1,2,3\}), \subseteq, \cup, \cap)$



(b) Set of all the divisors of 60 ordered by the divisibility relation



(c) $(\mathcal{P}(\{1,2,3\}) \setminus \{1,2,3\}, \subseteq, \cup, \cap)$



(d) $(\mathcal{P}(\{1,2,3\}) \setminus \{1\}, \subseteq, \cup, \cap)$

Figure 2.1. *Examples of partially ordered sets represented with Hasse diagram.*

and a greatest element $\{1,2,3\}$. Hence, it is a complete lattice. For instance, the pair $\{\{1,2\},\{1,3\}\}$ has a least upper bound $\{1,2\} \cap \{1,3\} = \{1\}$ and a greatest lower bound $\{1,2\} \cup \{1,3\} = \{1,2,3\}$.

The second one (Figure 2.1(b)) corresponds to the set of the divisors of 60: $\{1,2,3,5,6,10,12,\ 15,20,30,60\}$ with the divisibility relation. Similarly, this partially ordered set has a least element 1 and a greatest element 60. Thus it is a complete lattice. The pair $\{3,4\}$ has a greatest lower bound 1 (their greatest common divisor) and a least upper bound 12 (their least common multiple).

On the contrary, the third example (Figure 2.1(c)) is not a lattice. Indeed, the pair $\{\{1,2\},\{1,3\}\}$ does not have a least upper bound. Likewise, if the element $\{\emptyset\}$ is removed from the lattice Figure 2.1(a), the partially ordered set obtained is no longer a lattice. However, removing any element that is neither the least nor the greatest element of the lattice does not change the fact that it is a lattice as shown on Figure 2.1(d).

Example 2.1.4 – It is easily verified that the partially ordered set $(\mathbb{I}^n, \subseteq, \cup, \cap)$ with the least element $\bot = \emptyset$ and the greatest element $\top = \mathbb{F}^n$ is a complete lattice. Let $I = I_1 \times \cdots \times I_n$ and $I' = I'_1 \times \cdots \times I'_n$ be any two elements of $\mathbb{I}^n$. The pair $\{I, I'\}$ has a *glb*

$$I \cap I' = [\max(\underline{I_1}, \underline{I'_1}), \min(\overline{I_1}, \overline{I'_1})] \times \cdots \times [\max(\underline{I_n}, \underline{I'_n}), \min(\overline{I_n}, \overline{I'_n})]$$

and a *lub*

$$I \cup I' = [\min(\underline{I_1}, \underline{I'_1}), \max(\overline{I_1}, \overline{I'_1})] \times \cdots \times [\min(\underline{I_n}, \underline{I'_n}), \max(\overline{I_n}, \overline{I'_n})]$$

It follows that any subset has a least upper bound and a greatest lower bound and therefore $(\mathbb{I}^n, \subseteq, \cup, \cap)$ is a lattice. Moreover, this lattice is finite, thus $(\mathbb{I}^n, \subseteq, \cup, \cap)$ is a complete lattice.

Lattices are the base set upon which rely the abstract domains in Abstract Interpretation. An important feature of the abstract domains is that they can be link by Galois connexion. Notice that some abstract domains do not have a Galois connexion (Remark 2.1.5). Galois connexion have been applied to the semantics by Cousot and Cousot in [COU 77a] as follows.

Definition 2.1.3 (Galois Connexion)**.** Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two partially ordered sets, a Galois Connexion is defined by two morphisms, an abstraction $\alpha : \mathcal{D}_1 \to \mathcal{D}_2$ and a concretization $\gamma : \mathcal{D}_2 \to \mathcal{D}_1$ such that:

$$\forall X_1 \in \mathcal{D}_1, X_2 \in \mathcal{D}_2, \alpha(X_1) \sqsubseteq X_2 \iff X_1 \sqsubseteq \gamma(X_2)$$

Galois connexion are usually reprsented as follows:

$$\mathcal{D}_1 \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathcal{D}_2$$

Remark 2.1.3 – An important consequence of this definition is that the function α and γ are monotonic for the order $\sqsubseteq$ [COU 92a], that is:

$$\forall X_1, Y_1 \in \mathcal{D}_1, X_1 \sqsubseteq Y_1 \Rightarrow \alpha(X_1) \sqsubseteq \alpha(Y_1), \text{ and}$$
$$\forall X_2, Y_2 \in \mathcal{D}_2, X_2 \sqsubseteq Y_2 \Rightarrow \gamma(X_2) \sqsubseteq \gamma(Y_2)$$

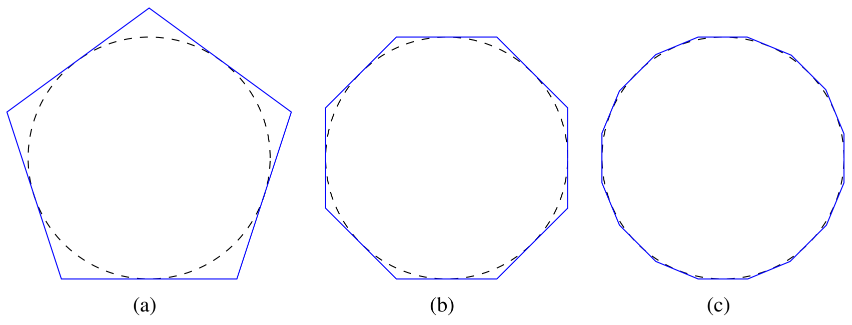


Figure 2.2. *Approximations for a circle*

Remark 2.1.4 – This definition implies that $(\alpha \circ \gamma)(X_2) \sqsubseteq X_2$ and $X_1 \sqsubseteq (\gamma \circ \alpha)(X_1)$. X_2 is said to be an approximation (or abstraction) correct of X_1.

Remark 2.1.5 – Note that for a given abstract domain there can be no abstraction function. For instance, the polyhedra abstract domain has no abstraction function. Indeed, there exists an infinity of approximations of a circle with a polyhedron. Therefore, there is no Galois connexion for the polyhedra abstract domain.

Figure 2.2 shows three different approximations for a circle using polyhedra. As there exists an infinity of tangent to the circle, there potentially exists a polyhedron with an infinite number of sides exactly approximating the circle.

Galois connexion are used in Constraint Programming even thought there are not named. For instance there are used when solving continuous problems. Indeed, as the intervals with real bounds are not computer representable, there are approximated with intervals with floating-point bounds. The transition from one representation to the other forms a Galois connexion as shown in the following example.

Example 2.1.5 – Let $\mathbb{J}$ be the set of intervals with real bounds. Given two partially ordered sets $(\mathbb{I}^n, \subset)$ and $(\mathbb{J}^n, \subset)$, there exists a Galois connexion:

$$\begin{tikzcd}
\mathbb{J}^n \arrow[r, shift right, "\alpha_{\mathbb{I}}"'] & \mathbb{I}^n \arrow[l, shift right, "\gamma_{\mathbb{I}}"']
\end{tikzcd}$$

$$\alpha_{\mathbb{I}}([x_1, y_1] \times \cdots \times [x_n, y_n]) = [\underline{x_1}, \overline{y_1}] \times \cdots \times [\underline{x_n}, \overline{y_n}]$$
$$\gamma_{\mathbb{I}}([a_1, b_1] \times \cdots \times [a_n, b_n]) = [a_1, b_1] \times \cdots \times [a_n, b_n]$$

In this example, the abstraction function $\alpha_{\mathbb{I}}$ transforms a Cartesian product of real bound intervals into a Cartesian product of floating-point bounds intervals. It approximates each real bounds by the closest floating-point number rounded in $\mathbb{F}$ in the corresponding direction. As for the concretization function, it is straightforward since a floating-point number is also a real.

2.1.2.2. *Concrete/Abstract*

The concrete domain, denoted by $\mathcal{D}^\flat$, corresponds to the values that can be taken by the variables throughout the program $\mathcal{D}^\flat = \mathcal{P}(V)$ with V a set. Computing the concrete domain can be undecidable, thus an approximation is accepted. The approximation of the concrete domain is called abstract domain and is denoted by $\mathcal{D}^\sharp$. If there exists a Galois connexion between the concrete domain and the abstract domain, $\mathcal{D}^\flat \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathcal{D}^\sharp$, then any concrete function $f^\flat$ in $\mathcal{D}^\flat$ as an abstraction $f^\sharp$ in $\mathcal{D}^\sharp$ such that

$$\forall X^\sharp \in \mathcal{D}^\sharp, (\alpha \circ f^\flat \circ \gamma)(X^\sharp) \sqsubseteq f^\sharp(X^\sharp)$$

This is a consequence of the Remark 2.1.4. Moreover, the abstract function $f^\sharp$ is said optimal if and only if $\alpha \circ f^\flat \circ \gamma = f^\sharp$.

In the following we will write $\mathcal{D}$ (resp. f) for a domain (resp. function) in general (wheter it is concrete or abstract). And we will write $\mathcal{D}^\flat$ and $f^\flat$ for a concrete domain and a concrete function, and $\mathcal{D}^\sharp$ and $f^\sharp$ for an abstract domain and an abstract function.

2.1.2.3. *Transfer Function*

In order to analyze a program, each line of code is analyzed. To do so, each instruction is associated to a function, called transfer function, which modifies the possible values for the variables.

Definition 2.1.4 (Transfer Function)**.** Let C be the line of code to analyze. Given an initial set of states, a transfer function $F : \mathcal{P}(\mathcal{D}) \rightarrow \mathcal{P}(\mathcal{D})$ returns a set of environments corresponding to all the possible accessible states after the execution of C. We will write $\{\!|C|\!\}$ the transfer function for the instruction C and $\{\!|C|\!\}X$ when it is applied to the environments set X.

Example 2.1.6 – Consider the affectation $x \leftarrow expr$ with x a variable and $expr$ any expression. The transfer function $\{\!|x \leftarrow expr|\!\}$ only modifies in the initial environments set the possible values for the variable x.

Let x and y be two variables taken their values in $[-10, 10]$. The transfer function $\{\!|x \leftarrow random(1, 10)|\!\}$ only modifies the values for x. We now have x in $[1, 10]$ and y in $[-10, 10]$.

Example 2.1.7 – Let us now consider a boolean expression, the transfer function only keeps the environments satisfying the boolean expression. Let x and y be two variables taken their values in $[-10, 10]$. For the boolean expression $x \leq 0$, the transfer function $\{\!|x \leq 0|\!\}$ filters the values of x so as to satisfy the boolean expression. We now have x in $[-10, 0]$ and y in $[-10, 10]$.

Remark 2.1.6 – In the following, any transfer function is supposed to be monotonic.

2.1.2.4. *Fixpoint*

Abstract Interpretation also rely on the notion of fixpoint.

Definition 2.1.5 (Fixpoint)**.** Let F be a function, we called fixpoint of F, an element X such that $F(X) = X$. We denote by $\mathrm{lfp}_X\, F$ a least fixpoint of F bigger than X, and $\mathrm{gfp}_X\, F$ a greatest fixpoint of F smaller than X.

Remark 2.1.7 – Note that when F is monotonic, if the least or the greatest fixpoint exists then it is unique.

Each instruction of the program is associated with a transfer function. The program thus corresponds to a composition of these functions. Proving that the program is correct is equivalent to computing the least fixpoint of this composition of functions. The computation of the fixpoint is mandatory to analyze loops for instance. The functions associated to the loop are applied several times until the fixpoint is reached.

There exists several possible iterative schemas. Let $(X_1, \dots, X_n)$ be the set of environments set, X_i corresponds to the environments set for the instruction i. We denote by X_i^j the set of environments for the instruction i at the iteration j. Let F_i be the transfer function for the instruction i. The most common iterative scheme is the Jacobi iterations one. The value of X_i^j is computed using the environments sets from the previous iteration:

$$X_i^j = F_i(X_1^{j-1}, \dots, X_n^{j-1})$$

An another iterative scheme is the Gauss-Seidel iterations one. It computes the value of X_i^j using the environments sets already computed at the current iteration and the environments sets from the previous iteration:

$$X_i^j = F_i(X_1^j, \dots, X_{i-1}^j, X_i^{j-1}, X_{i+1}^{j-1}, \dots X_n^{j-1})$$

In the examples of this section, we use the Gauss-Seidel iterations.

Example 2.1.8 – Consider the following program:

```
1: int $x, y$
2: $x \leftarrow 0$
3: $y \leftarrow x$
4: while $x < 10$ do
5:     $y \leftarrow 2x$
6:     $x \leftarrow x + 1$
7: end while
```

We have:

$$\begin{aligned}
&X_1 = \top\\
&X_2 = \{\!|x \leftarrow 0|\!\}X_1\\
&X_3 = \{\!|y \leftarrow x|\!\}X_2\\
&X_4 = X_3 \cup X_6\\
&X_{4'} = \{\!|x < 10|\!\}X_4\\
&X_5 = \{\!|y \leftarrow 2x|\!\}X_{4'}\\
&X_6 = \{\!|x \leftarrow x+1|\!\}X_5\\
&X_7 = \{\!|x \geq 10|\!\}X_4
\end{aligned}$$

with $X_{4'}$ the set of environments if the condition of the **while** loop is satisfied. All the environments set are initialized at $\bot = \emptyset$, except the first one which is initialized to $\top = \mathbb{Z}^2$.

By applying a first time all the transfer functions, we obtain:

$$\begin{aligned}
&X_1 = \top\\
&X_2 = \{x = 0, y \in \mathbb{Z}\}\\
&X_3 = \{x = y = 0\}\\
&X_4 = \{x = y = 0\}\\
&X_{4'} = \{x = y = 0\}\\
&X_5 = \{x = y = 0\}\\
&X_6 = \{x = 1, y = 0\}\\
&X_7 = \bot
\end{aligned}$$

By applying a second time all the transfer functions, we obtain:

$$\begin{aligned}
&X_1 = \top\\
&X_2 = \{x = 0, y \in \mathbb{Z}\}\\
&X_3 = \{x = y = 0\}\\
&X_4 = \{x \in [\![0,1]\!], y = 0\}\\
&X_{4'} = \{x \in [\![0,1]\!], y = 0\}\\
&X_5 = \{x \in [\![0,1]\!], y \in [\![0,2]\!]\}\\
&X_6 = \{x \in [\![1,2]\!], y \in [\![0,2]\!]\}\\
&X_7 = \bot
\end{aligned}$$

where $[\![a,b]\!] = \{x \in \mathbb{Z} \,|\, a \leq x \leq b\}$ the intervals of integers between a and b.

The sets of environments X_2 and X_3 have not been modified and depend on environments sets that have not been modified either. We thus said that their fixpoint is reached and their transfer functions are not applied anymore. On the contrary, the set of environments X_7 have not been modified but depends on environments sets that have been modified (X_4). The other transfer functions are applied until the fixpoint is reached. After twelve iterations, the fixpoint is reached. We thus have:

$X_1 = \top$
$X_2 = \{x = 0, y \in \mathbb{Z}\}$
$X_3 = \{x = y = 0\}$
$X_4 = \{x \in [\![0, 10]\!], y \in [\![0, 18]\!]\}$
$X_{4'} = \{x \in [\![0, 9]\!], y \in [\![0, 18]\!]\}$
$X_5 = \{x \in [\![0, 9]\!], y \in [\![0, 18]\!]\}$
$X_6 = \{x \in [\![1, 10]\!], y \in [\![0, 18]\!]\}$
$X_7 = \{x = 10, y \in [\![0, 18]\!]\}$

We can thus say that throughout this program, the variable x is between 0 and 10 and that the variable y is between 0 and 18. Furthermore, after the execution of this program, the variable x is equal to 10 and the variable y is between 0 and 18. This results is an approximation of the possible values, indeed, at the end of the program execution, y is always equal to 18.

Analyzing a program is equivalent to over-approximating the value of the environment at any point in the program. Starting from the least element $\perp$, the successive application of the transfer functions allows the fixpoint to be computed. This fixpoint corresponds to the least fixpoint of the composition of the transfer functions bigger than $\perp$. We can say that in Abstract Interpretation, computing the least fixpoint $\mathrm{lfp}_\perp F$ with $F : \mathcal{D} \to \mathcal{D}$ the composition of the transfer functions and $\mathcal{D}$ a complete partially ordered set or lattice, is equivalent to analyzing the program.

However, the least fixpoint may not be reached when its computation is undecidable. In this case, an approximation is computed. Additionally, the computation of the fixpoint can converge very slowly, especially when analyzing a loop. To accelerate this process, a widening operator is used.

Definition 2.1.6 (Widening)**.** The binary abstract operator $\triangledown^\sharp$ from $\mathcal{D}^\sharp \times \mathcal{D}^\sharp$ to $\mathcal{D}^\sharp$ is a widening if and only if:

1) $\forall X^\sharp, Y^\sharp \in \mathcal{D}^\sharp, (X^\sharp \triangledown^\sharp Y^\sharp) \sqsupseteq^\sharp X^\sharp, Y^\sharp$, and

2) for any chain $(X_i^\sharp)_{i\in\mathbb{N}}$, the increasing chain $(Y_i^\sharp)_{i\in\mathbb{N}}$ defined by:

$$\begin{cases} Y_0^\sharp & = X_0^\sharp \\ Y_{i+1}^\sharp & = Y_i^\sharp \triangledown^\sharp X_{i+1}^\sharp \end{cases}$$

is stable after a finite number of iterations, *i.e.,* $\exists K$ such that $Y_{K+1}^\sharp = Y_K^\sharp$.

The widening operator allows the lest fixpoint lfp to be computed. It computes, in function of several previous iterations, an approximation of the least fixpoint by staying above it in the lattice. This operator performs increasing iterations, that is iterations that make the set of possible values for the variables grows.

Remark 2.1.8 – For any partially ordered set, the widening computes an approximation of the least fixpoint in a finite number of iterations. In particular, when the considered partially ordered set has an infinite increasing chain, the widening is required to reach an approximation of the least fixpoint in a finite number of iterations. It ensures the termination of the analysis.

Remark 2.1.9 – Notice that if the widening and the transfer functions are correct, then the correction of the analysis is ensured.

Remark 2.1.10 – If the result obtain with the widening meet the specifications, then the program is correct. In fact, if the set of environments X meets the specifications then a smaller set of environments $X' \sqsubseteq X$ also meets them. However, nothing can be deduced if the result obtain with the widening does not meet the specifications.

Example 2.1.9 – Let us consider the program in Example 2.1.8. This time we use a widening operator to analyze the **while** loop. We thus redefine the environment corresponding to the loop condition:

$$X_4 = X_4 \triangledown^\sharp (X_3 \cup X_6)$$

with $\triangledown^\sharp$ a widening operator such that:

$$\llbracket a, b \rrbracket \triangledown^\sharp \llbracket c, d \rrbracket = \left\llbracket \begin{cases} a & \text{if } a \leq c \\ -\infty & \text{otherwise} \end{cases}, \begin{cases} b & \text{if } b \geq d \\ +\infty & \text{otherwise} \end{cases} \right\rrbracket \text{ and,}$$
$$X \triangledown^\sharp \bot = X \text{ and } \bot \triangledown^\sharp X = X$$

This operator sets the upper bound for the variable x to $+\infty$ if between two iterations the interval of the variable x grows toward $+\infty$. Conversely, the lower bound is set to $-\infty$ if the interval for the variable decreases.

Let's look at the evolution of the environment for X_4. After the first iteration, we have:

$$\begin{aligned} X_4 &= \bot \triangledown^\sharp \{x = y = 0\} \\ &= \{x = y = 0\} \end{aligned}$$

Which corresponds to the result obtained after the first iteration in Example 2.1.8. After the second iteration, we obtain:

$$\begin{aligned} X_4 &= \{x = y = 0\} \triangledown^\sharp \{x \in \llbracket 0, 1 \rrbracket, y = 0\} \\ &= \{x \in \llbracket 0, +\infty \rrbracket, y = 0\} \end{aligned}$$

The widening operator deduces from the previous iteration that the interval for x grows in the loop et thus modifies its upper bound. Similarly, at the following iteration, the widening deduces that the interval for y grows in the loop and modifies its upper bound. After two iterations, the fixpoint is reached. We now have:

$$\begin{aligned}
&X_1 = \top\\
&X_2 = \{x = 0, y \in \mathbb{Z}\}\\
&X_3 = \{x = y = 0\}\\
&X_4 = \{x \in [\![0, +\infty]\!], y \in [\![0, +\infty]\!]\}\\
&X_{4'} = \{x \in [\![0, 9]\!], y \in [\![0, +\infty]\!]\}\\
&X_5 = \{x \in [\![0, 9]\!], y \in [\![0, 18]\!]\}\\
&X_6 = \{x \in [\![1, 10]\!], y \in [\![0, 18]\!]\}\\
&X_7 = \{x \in [\![10, +\infty]\!], y \in [\![0, +\infty]\!]\}
\end{aligned}$$

We can thus say that after the execution of instruction 7, the variable x is greater or equal to 10 and that the variable y is positive. This is less precise than the result obtained in Example 2.1.8. However, only four iterations were required to reach the fixpoint, which is three times less than without using the widening.

Remark 2.1.11 – Note that the number of iterations required to reach the fixpoint with the widening operator does not depend on the constants in the program. By replacing the constant 10 by 100 or 1000, the iterations without the widening will converge more slowly, while those with the widening will not be affected by this change.

This operator can generate a very large over-approximation of the fixpoint, as shown in Example 2.1.9 where the only information that we obtained for y is that it is positive while in Example 2.1.8 we were able to infer that y was less or equal to 18. Similarly, the variable x is greater or equal to 10 while in Example 2.1.8 it was equal to 10. In order to refine this approximation, a narrowing operator can be used.

Definition 2.1.7 (Narrowing)**.** The abstract binary operator $\triangle^\sharp$ from $\mathcal{D}^\sharp \times \mathcal{D}^\sharp$ to $\mathcal{D}^\sharp$ is a narrowing if and only if:

1) $\forall X^\sharp, Y^\sharp \in \mathcal{D}^\sharp, (X^\sharp \sqcap^\sharp Y^\sharp) \sqsubseteq^\sharp (X^\sharp \triangle^\sharp Y^\sharp) \sqsubseteq^\sharp X^\sharp$, and

2) for any chain $(X_i^\sharp)_{i\in\mathbb{N}}$, the chain $(Y_i^\sharp)_{i\in\mathbb{N}}$ defined by:

$$\begin{cases} Y_0^\sharp &= X_0^\sharp \\ Y_{i+1}^\sharp &= Y_i^\sharp \triangle^\sharp X_{i+1}^\sharp \end{cases}$$

is stable after a finite number of iterations, *i.e.,* $\exists K$ such that $Y_{K+1}^\sharp = Y_K^\sharp$.

Like the widening, the narrowing deduces form the previous iterations an over-approximation of a fixpoint. However the computed fixpoint is not necessary the least fixpoint lfp. Indeed, the narrowing operator computes an approximation of a fixpoint (gfp or lfp) while remaining above it. Unlike the widening, the narrowing performs decreasing iterations, that is iterations that make the set of possible values for the variables lessens.

Example 2.1.10 – Let us consider the program in Example 2.1.8. In the previous example (Example 2.1.9), we used a widening operator in order to speed up the computation of the fixpoint. Using this operator generated a large over-approximation. Starting from the result obtained in Example 2.1.9, we use a narrowing operator to refine this result. The environment corresponding to the loop condition is thus redefined as:

$$X_4 = X_4 \vartriangle^\sharp (X_3 \cup X_6)$$

with $\vartriangle^\sharp$ a narrowing operator such that:

$$[\![a, b]\!] \vartriangle^\sharp [\![c, d]\!] = \left[\!\!\left[\begin{cases} c & \text{if } a = -\infty \\ a & \text{otherwise} \end{cases}, \begin{cases} d & \text{if } b = +\infty \\ b & \text{otherwise} \end{cases} \right]\!\!\right] \text{ and,}$$
$$X \vartriangle^\sharp \bot = \bot \text{ and } \bot \vartriangle^\sharp X = \bot$$

This operator only reduces the infinity bounds. The bounds are thus reduced at most one time.

Let us see the evolution of the environment X_4, after the first iteration, we have:

$$\begin{aligned} X_4 &= \{x \in [\![0, +\infty]\!], y \in [\![0, +\infty]\!]\} \vartriangle^\sharp \{x \in [\![0, 10]\!], y \in [\![0, 18]\!]\} \\ &= \{x \in [\![0, 10]\!], y \in [\![0, 18]\!]\} \end{aligned}$$

Which correspond to the result obtained after eleven iteration in the Example 2.1.8. After only two iterations, the fixpoint is reached and is the same as in Example 2.1.8. By using the widening and the narrowing operators, the fixpoint is reached in six iterations, which is two times less that without using them. Note that in this example the reached fixpoint is the same in both case. In general, if a fixpoint is reached in a finite number of iterations without using a widening and a narrowing operator then this fixpoint may be more precise than the one obtained using these two operators.

Notice that starting from Example 2.1.9, applying the transfer functions as defined in Example 2.1.8 generates decreasing iterations. These decreasing iterations reduce the environments without using the narrowing operator.

The schema of an approximation of a fixpoint using a widening and a narrowing operator is given Figure 2.3. Starting from the least element $\bot$, two increasing iterations are performed. The set of possible values for the variables grows. Then a widening operator $\triangledown^\sharp$ is used making the environment go above the least fixpoint lfp. Note that the widening may make the environment go above the greatest fixpoint gfp. The narrowing operator $\vartriangle^\sharp$ is then used in order to refine the obtained approximation. This decreasing iteration refine the approximation while staying above of the considered fixpoint, which in our case is the least fixpoint lfp. To sum up, starting from an environment under the desired fixpoint the widening operator makes the set of variables values grows allowing a point above the fixpoint to be reached. As for

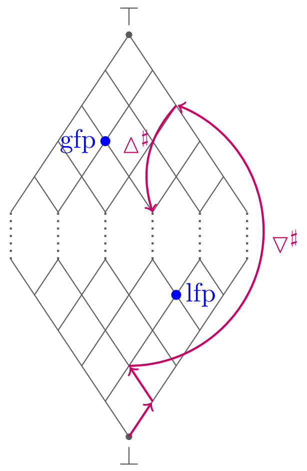


Figure 2.3. *Schema of an approximation of the least fixpoint using a widening and a narrowing operator.*

the narrowing, given an environment above a fixpoint, it reduces the set of variables values while staying above the considered fixpoint.

While a lot of work has been devoted to designing smart widenings [BAG 05a, BAG 05b, BAG 06, SIM 06, D'S 06, COR 08, MON 09, SIM 10], narrowings have gathered far less attention [COU 92b, MOR 93, ALP 93, HIC 97, COR 11]. Some major domains, such as polyhedra, do not feature any. This difference of interest between the widening and the narrowing may be explained by three facts: firstly, narrowings are not necessary to achieve soundness unlike widenings which are mandatory. Indeed, the widening allows an approximation of the least fixpoint to be computed even if the partially ordered set has an infinite increasing chain. So if the result obtained with the widening meets the given specifications then the considered program is sound. However if the result obtained with the widening does not meet the specifications, this does not mean that the program is not sound. This can be due to a toot large over-approximation. Therefore, it is mandatory for the widening to be properly designed to minimize the over-approximation and thus avoid false alarms.

Secondly, applying the transfer function without widening and narrowing operators may performs a bounded number of decreasing iterations and is sometimes sufficient to recover enough precision after widening [BER 10]. This is the case for the program used to illustrate this section (Example 2.1.8). After the widening, the use of the transfer functions without widening reduces the environments leading to a more precise approximation of the least fixpoint. In this example, the resulting fixpoint is the same as the one obtained with the narrowing. Moreover, the termination is guaranteed as the number of decreasing iterations is finite and generally fixed in advance.

Finally, when this simple technique is not sufficient, narrowing operators do not actually help further in practice to improve the accuracy and other methods beyond decreasing iterations must be considered such as the one presented by Halbwachs and Henry in [HAL 12]. After a series of both increasing and decreasing iterations, the fixpoint is reached, however the achieved precision is not satisfactory. In this article, the authors propose to apply a series of iterations and keep only a portion of the previous result. In other words, to keep for each environment only the possible values for a subset of variables. The performed experiments show a reduction of the over-approximation generated by the widening operator.

Nevertheless, when the over-approximation is due to imprecise transfer functions, another solution is to used local iterations introduced by Granger in 1992 [GRA 92]. A series of decreasing iterations is applied several times. This method is detailed in the next section.

2.1.2.5. *Local iterations*

The abstract transfer function $F^\sharp \sqsupseteq (\alpha \circ F^\flat \circ \gamma)$ does not always compute efficiently the smallest abstract domain containing the considered expression, even though when it is optimal $(\gamma \circ F^\sharp \circ \alpha) = F^\flat$. To efficiently compute the result of the transfer functions, transfer functions often involve lower closure operators [GRA 92].

Definition 2.1.8 (Lower Closure Operator)**.** An operator $\rho : \mathcal{D} \rightarrow \mathcal{D}$ is a lower cloture operator if and only if ρ is:

1) monotonic, $\forall X, Y \in \mathcal{D}, X \sqsubseteq Y \Rightarrow \rho(X) \sqsubseteq \rho(Y)$

2) reductive, $\forall X \in \mathcal{D}, \rho(X) \sqsubseteq X$, and

3) idempotent, $\forall X \in \mathcal{D}, \rho(X) = (\rho \circ \rho)X$.

Granger's work [GRA 92] shows that by iterating several times lower closure operators the approximation of the fixpoint may be improved. Given a correct abstraction $\rho^\sharp$ of $\rho^\flat$, the limit $Y_\delta^\sharp$ of the sequence of the narrowing operator, $Y_0^\sharp = X^\sharp$, $Y_{i+1}^\sharp = Y_i^\sharp \vartriangle \rho^\sharp(Y_i^\sharp)$ is an abstraction of $(\rho^\flat \circ \gamma)(X^\sharp)$. Note that even if $\rho^\sharp$ is not an optimal abstraction of $\rho^\flat$, $Y_\delta^\sharp$ may be significantly more precise than $\rho^\sharp(X^\sharp)$. A relevant application is the analysis of complex test conjunction $C_1 \wedge \cdots \wedge C_p$ where each atomic test C_i is modeled in the abstract as $\rho_i^\sharp$. Generally, $\rho^\sharp = \rho_1^\sharp \circ \cdots \circ \rho_p^\sharp$ is not optimal, even when each $\rho_i^\sharp$ is.

Lower closure operators may be reformulated as a fixpoint. This unifies the use of narrowings and brings out the similarities in the iterative computations. Given an element X, ρ computes the greatest fixpoint smaller than X, that is $\rho(X) = \mathrm{gfp}_X\, \rho$. Local iterations may be used at any time in the analysis and not only after a widening.

Given a program to analyze, an abstract domain is chosen to best represent the program properties. Ther exists several types of abstract domains. A short presentation of abstract domains is given in the next subsection.

2.1.2.6. *Abstract Domains*

Abstract domains play a key role in Abstract Interpretation. Because of the importance of numerical properties and variables in a program, many numerical abstract domains are developed. Major ones include intervals [COU 77a] and polyhedra [COU 78]. Recent years have seen the development of many new abstract domains capturing other properties, such as octagons [MIN 06], ellipsoids [FER 04], octahedra [CLA 04] and even varieties [ROD 04]. In addition, support libraries for abstract domains such as Apron [JEA 09] were designed. These new domains can handle all kinds of numeric variables, mathematical integers, rationals, reals, machine integers and floating-point numbers [MIN 12]. They even express relationships between variables of different types [MIN 04], and between numeric and boolean variables [BER 10]. Moreover, generic operators can be used to build new domains from existing abstract domains such as disjunctive completion [COU 92a], reduced products [COU 79, COU 07, COU 11] and partitioning [BOU 92, RIV 07]. These abstract domains are bricks in static analyzers, which generally use different abstract domains simultaneously. It is crucial to carefully choose (or design) abstract domains for a given problem. This is usually done by hand based on the properties that must be deducted.

These various representations are grouped according to their expressiveness. The more expressible properties by an abstract domain are complex, the more it will be accurate. Expressible properties by an abstract domain corresponds to the relations between different variables that can be represented by the abstract domain. In other words, the more representable relations between variables are complex, the more the abstract domain is accurate. This precision usually comes at a cost in terms of computation time. There are three main categories: the non-relational, relational and weakly relational domains. In the first family, the properties are expressed on a single variable, the abstract domains of this family are the least expressive. To represent several variables, we use a Cartesian product of the selected domain. The best known in this area is the intervals abstract domains [COU 76] where each variable is represented with an interval of possible values for this variable.

In the other two families of abstract domains, we can, as their name suggests, have relations between variables. Relational domains are very expressive. There exist a large diversity of these abstract domains. For instance, there is the polyhedra abstract domains [COU 78], which expresses linear relationships between variables,

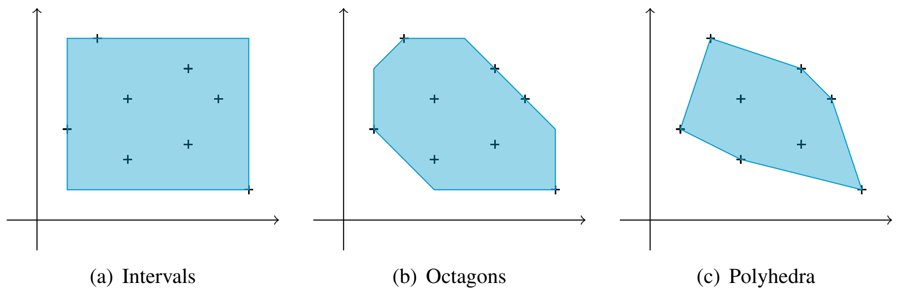


Figure 2.4. *Different abstract domains representing the same set of points*

and the ellipsoids abstract domain [FER 04], expressing second degree polynomials of ellipses. The last family, the weakly relational domains is a tradeoff between the two families named above. It is composed of abstract domains offering a tradeoff between accuracy and computation time. These weakly relational domains can represent some of the possible expressible properties between variables. They were introduced in 2004 by Miné [MIN 04]. Among these abstract domains there exists the zone abstract domain expressing inequalities of the form $v_1 - v_2 \leq c$ with v_1 and v_2 variables and c a constant, and the octagon abstract domain expressing inequalities of the form $\pm v_1 \pm v_2 \leq c$.

Figure 2.4 shows for a same set of points an example of abstract domain of each of the three categories previously named. The first one corresponds to a non-relational domain, the intervals (2.4(a)), the second to a weakly relation domain, the octagons (2.4(b)), and the last one to a relational one, the polyhedra (2.4(c)). We can see that the more the properties representable by an abstract domain are complex, the more the abstract domain is accurate. The polyhedra are more precise than the octagons which are more precise than the intervals.

An abstract domain corresponds to a computable set $\mathcal{D}^\sharp$ with the partial order $\sqsubseteq^\sharp$ to handle computable elements. Therefore, every operators in $\mathcal{D}^\flat$ must have an abstraction in $\mathcal{D}^\sharp$. These different operators are listed below.

Operators on abstract domains:

– a concretization function $\gamma : \mathcal{D}^\sharp \to \mathcal{D}^\flat$, and if it exists an abstraction function $\alpha : \mathcal{D}^\flat \to \mathcal{D}^\sharp$ forming a Galois connexion $\mathcal{D}^\flat \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathcal{D}^\sharp$,

– a least element $\bot^\sharp$ and a greatest element $\top^\sharp$ such that $\gamma(\bot^\sharp) = \emptyset$ et $\gamma(\top^\sharp) = V$ with $\mathcal{D}^\flat = \mathcal{P}(V)$,

– efficient algorithms to compute transfer functions,

– efficient algorithms for the meet $\cup^\sharp$ and join $\cap^\sharp$,

– efficient algorithms for the widening $\nabla^\sharp$ if $\mathcal{D}^\sharp$ has an infinite increasing chain, and

– if it exists and $\mathcal{D}^\sharp$ has an infinite decreasing chain, efficient algorithms for the narroxing $\triangle^\sharp$.

Note that there may be no abstraction function et no narrowing operator, and, although relatively rare, there can be no join. As mentioned Remark 2.1.5, the polyhedra abstract domain has no abstraction function and therefore no Galois connexion. Moreover, it does not feature any narrowing operator. Although they may not exist, these different operators are useful in order to have only one possible representation for a given concrete domain (abstraction) and to refine the over-approximation while computing the fixpoint (narrowing).

2.1.3. *Conclusion*

The previous sections have briefly presented Abstract Interpretation. The main definitions and notions needed in the following are given. For a more detailed presentation the reader is advised to read [COU 77a] for instance.

To sum up, Abstract Interpretation automatically analyzes programs in order to certify that they do not contain any execution errors. This analyze relies on the computation of the program semantic, the set of all the possible values for the variables of the program. Directly computing the concrete semantic can be very consuming in terms of computation time and even be undecidable. Thus the semantic is abstracted. In other words, only some characteristics are kept in order to simplify and speed up the process while keeping the process sound. For each concrete operators, an abstract operator is defined and efficient algorithms are designed. The abstract semantic is approximated using an abstract domain. There exists several abstract domains in Abstract Interpretation offering different tradeoff between precision and computation time.

After this survey of Abstract Interpretation general theory and techniques, the next section briefly presents Constraint Programming, its principles, scientific challenges and resolution techniques.

2.2. Constraint Programming

Constraint Programming (CP) has been introduced in 1974 by Montanari [MON 74]. It relies on the idea that combinatorial problems can be expressed as a conjunction of first-order logic formulas, called constraints. A problem is said combinatorial when it contains a very large number of combinations of possible values. Take the example of a Sudoku puzzle: each cell that does not already have a value can take the value between 1 and 9. The number of all the possible grids, corresponding to the enumeration of all the possible values for all the empty boxes, is very large. If only 10 cells are empty, this number is larger than 3 million. It is equal to 9^k where k is the number of empty cells as each empty cell has 9 possible values. This is a combinatorial problem. Listing all these grids to check if they correspond to a solution would take too long. Constraint Programming implements techniques to efficiently solve such combinatorial problems. As described by E. Freuder, in Constraint Programming, the user specifies the problem and the computer solves it:

> *"Constraint Programming represents one of the closest approaches computer science has yet made to the Holy Grail of programming: the user states the problem, the computer solves it."*
>
> – Eugene C. Freuder [FRE 97].

In order to define the problem, the user defines the constraints, that are the specifications of the problem. A constraint represents a specific combinatorial relationship of the problem [ROS 06]. Still with the example of Sudoku, the fact that each number between 1 and 9 appears exactly once in each row is a constraint. Each constraint comes with *ad hoc* operators using the structure expressed by the constraint to reduce combinatorics. The constraints are then combined into solving algorithms. Most research efforts in Constraint Programming focuses on defining and improving constraints [FAG 11], developing efficient algorithms [BES 11, PET 11] or fine tuning for the solvers [ANS 09, ARB 09]. There are many global constraints [BES 03, BEL 10] and each comes with an algorithm reducing its combinatorics.

Constraint Programming provides effective techniques for combinatorial resolution with many real-life applications, among which are scheduling [GRI 11, HER 11a], cryptography [RAM 11], rostering [STØ 11], music [TRU 11], biology [PER 09], or network design problems [PEL 09].

However, there are still limitations to the use of Constraint Programming, one of the most important being the lack of solving algorithms capable of handling both discrete and continuous variables. In 2009, works [BER 09, CHA 09b] have proposed methods to solve mixed problems, however, the techniques used transform the variables in order to only have discrete or continuous variables.

Finally, Constraint Programming offers only limited choices of variables representations which are integer or real Cartesian product of a set of base.

In the following we present the basics of Constraint Programming necessary for the understanding of this book. We will focus only on the aspects that interest us. For a more detailed presentation, see [ROS 06].

2.2.1. *Principles*

In Constraint Programming, the problems are modeled in a specific format, as a Constraint Satisfaction Problem (CSP). Variables can either be integer or real.

Definition 2.2.1 (Constraint Satisfaction Problem)**.** A constraint satisfaction problem (CSP) is defined by a set of variables $(v_1 \dots v_n)$ taking values in domains $(\hat{D}_1 \dots \hat{D}_n)$ and a set of constraints $(C_1 \dots C_p)$. A constraint is a relation on a subset of variables.

Domain D_i is the set of possible values for the variable v_i. The set of all possible assignments for the variables $D = D_1 \times \cdots \times D_n$ is called *search space*. The search space is modified throughout the resolution, we note the initial search space $\hat{D} = \hat{D}_1 \times \cdots \times \hat{D}_n$. Problems can either be discrete ($\hat{D} \subseteq \mathbb{Z}^n$) or continuous ($\hat{D} \subseteq \mathbb{R}^n$) . Domains are always bounded in $\mathbb{R}$ or $\mathbb{Z}$.

The CSP solutions are the elements of $\hat{D}$ satisfying the constraints. We denote by S the solution set $S = \{(s_1 \dots s_n) \in \hat{D} \,|\, \forall i \in [\![1, p]\!],\, C_i(s_1 \dots s_n)\}$. For a constraint C, we note $S_C = \{(s_1 \dots s_n) \in \hat{D} \,|\, C(s_1 \dots s_n)\}$ the solution set for C

Example 2.2.1 – Let's consider the following 4×4 Sudoku grid:

	3	1	
			4
1	2		
			1

A possible model is to associate to each cell a variable as follows:

v_1	v_2	v_3	v_4
v_5	v_6	v_7	v_8
v_9	v_{10}	v_{11}	v_{12}
v_{13}	v_{14}	v_{15}	v_{16}

Each variable can take a value between 1 and 4. We thus have $\hat{D}_1 = \hat{D}_2 = \cdots = \hat{D}_{16} = [\![1, 4]\!]$. To specify that a cell has a fixed value, one can either modify its domain $\hat{D}_3 = \{1\}$, or add the constraint $v_3 = 1$. In a Sudoku each row, each column and each block must contains exactly once each value. For the first row, we thus have $v_1 \neq v_2, v_1 \neq v_3, v_1 \neq v_4, v_2 \neq v_3, v_2 \neq v_4,$ and $v_3 \neq v_4$. Specifying that all the variables of a subset have to be different can be done using the global constraint alldifferent: for the first row, we can replace the six constraints listed above by the constraint $\text{alldifferent}(v_1, v_2, v_3, v_4)$.

We obtain a CSP with sixteen variables $(v_1 \ldots v_{16})$ taking their values in the domains $\hat{D}_2 = \{3\}, \hat{D}_3 = \hat{D}_9 = \hat{D}_{16} = \{1\}, \hat{D}_8 = \{4\}, \hat{D}_{10} = \{2\}$, all the other domains being equal to $[\![1, 4]\!]$, and twelve constraints, $(C_1 \ldots C_4)$ corresponding to the rows, $(C_5 \ldots C_8)$ to the columns, and $(C_9 \ldots C_{12})$ to the blocks:

C_1 : $\text{alldifferent}(v_1, v_2, v_3, v_4)$

C_2 : $\text{alldifferent}(v_5, v_6, v_7, v_8)$

C_3 : $\text{alldifferent}(v_9, v_{10}, v_{11}, v_{12})$

C_4 : $\text{alldifferent}(v_{13}, v_{14}, v_{15}, v_{16})$

C_5: $\text{alldifferent}(v_1, v_5, v_9, v_{13})$

C_6: $\text{alldifferent}(v_2, v_6, v_{10}, v_{14})$

C_7: $\text{alldifferent}(v_3, v_7, v_{11}, v_{15})$

C_8: $\text{alldifferent}(v_4, v_8, v_{12}, v_{16})$

C_9 : $\text{alldifferent}(v_1, v_2, v_5, v_6)$

C_{10}: $\text{alldifferent}(v_3, v_4, v_7, v_8)$

C_{11}: $\text{alldifferent}(v_9, v_{10}, v_{13}, v_{14})$

C_{12}: $\text{alldifferent}(v_{11}, v_{12}, v_{15}, v_{16})$

Solutions of this CSP correspond to all the Sudoku grids properly filled. In this example there is one unique solution:

4	3	1	2
2	1	3	4
1	2	4	3
3	4	2	1

2.2.1.1. *Domains Representation*

Depending on the variables type, the domain may be stored as a range or a set of points. In the case of an integer variable v_i, the domain can be represented as a subset of points D_i or as a sub-interval of D_i.

Definition 2.2.2 (Integer Cartesian Product)**.** Let $v_1, \ldots, v_n$ be variables with discrete finite domains $\hat{D}_1, \ldots, \hat{D}_n$. We call *integer Cartesian product* any Cartesian product of any integer set in $\hat{D}$. Integer Cartesian products of $\hat{D}$ form a finite lattice for the inclusion:

$$\mathbb{S} = \left\{ \prod_i X_i \mid \forall i, X_i \subseteq \hat{D}_i \right\}$$

Definition 2.2.3 (Integer Box)**.** Let $v_1, \ldots, v_n$ be variables with discrete finite domains $\hat{D}_1, \ldots, \hat{D}_n$. We call *integer box* any Cartesian product of integer intervals in $\hat{D}$. Integer boxes of $\hat{D}$ form a finite lattice for the inclusion:

$$\mathbb{IB} = \left\{ \prod_i [\![a_i, b_i]\!] \,|\, \forall i, [\![a_i, b_i]\!] \subseteq \hat{D}_i, a_i \leq b_i \right\} \cup \emptyset$$

For real variables, as the reals are not computer representable, their domains are represented by floating bounds intervals.

Definition 2.2.4 (Box)**.** Let $v_1, \ldots, v_n$ be variables with bounded continuous domains $\hat{D}_1, \ldots, \hat{D}_n \in \mathbb{I}$. We call *box* any Cartesian product of floating bounds intervals in $\hat{D}$. The boxes in $\hat{D}$ form a finite lattice for the inclusion:

$$\mathbb{B} = \left\{ \prod_i I_i \,|\, \forall i, I_i \in \mathbb{I}, I_i \subseteq \hat{D}_i \right\} \cup \emptyset$$

2.2.1.2. *Constraint Satisfaction*

For integer variables, given an instantiation for the variables, a constraint answers *true* if this variables assignment satisfies the constraint and *false* otherwise.

Example 2.2.2 – Let v_1 and v_2 be two integer variables with domains $D_1 = [\![0, 2]\!]$, $D_2 = [\![0, 2]\!]$. Let $C : v_1 + v_2 \leq 3$ be a constraint. Given the assignment of the variable v_1 at 2 and of the variable v_2 at 0, the constraint C answers true, *i.e.* $C(2, 0)$ is true. In contrast, $C(2, 2)$ is false.

In the case of real domains, an important feature is that constraints can answer:

– *true*, if the box only contains solutions,

– *false*, if the box contains no solution at all,

– and *maybe*, when one cannot determine if the box contains solutions. This can happen when a box contains both solutions and non solutions elements

These different answers are due to the interval arithmetic [MOO 66]. In order to know whether a constraint is satisfied, each variable is replaced by its domain. Then the constraint is evaluated given the rules of interval arithmetic.

Let $I_1, I_2 \in \mathbb{I}$ be two intervals such that $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$, we have the following formulas:

$$
\begin{aligned}
I_1 + I_2 &= \left[\, \underline{a_1 + a_2},\ \overline{b_1 + b_2}\,\right] \\
I_1 - I_2 &= \left[\, \underline{a_1 - b_2},\ \overline{b_1 - a_2}\,\right] \\
I_1 \times I_2 &= [\ \min\left(\underline{a_1 \times a_2}, \underline{a_1 \times b_2}, \underline{b_1 \times a_2}, \underline{b_1 \times b_2}\right), \\
&\qquad \max\left(\overline{a_1 \times a_2}, \overline{a_1 \times b_2}, \overline{b_1 \times a_2}, \overline{b_1 \times b_2}\right)\ ] \\
I_1/I_2 &= I_1 \times [1/a_2, 1/b_2] \text{ if } 0 \notin I_2 \\
I_1^2 &= \begin{cases} \left[\min\left(\underline{a_1^2}, \underline{b_1^2}\right), \max\left(\overline{a_1^2}, \overline{b_1^2}\right)\right] & \text{if } 0 \notin I_1 \\ \left[0, \max\left(\overline{a_1^2}, \overline{b_1^2}\right)\right] & \text{otherwise} \end{cases}
\end{aligned}
$$

Example 2.2.3 – Let v_1 and v_2 be two continuous variables with domains $D_1 = [0, 2]$ et $D_2 = [0, 2]$. Consider the three following constraints:

C_1: $v_1 + v_2 \leq 6$
C_2: $v_1 - v_2 \geq 4$
C_3: $v_1 - v_2 = 0$

Given the variables domains, the first constraint C_1 answers *true*. Indeed, by replacing v_1 and v_2 by their domains, we have $[0, 2] + [0, 2] = [0, 4]$, and so $[0, 4] \leq 6$ which is true since any point of the interval $[0, 4]$ is less than or equal to 6. The second constraint C_2 answers *false*, indeed $[0, 2] - [0, 2] = [-2, 2]$ and the condition $[-2, 2] \geq 4$ is always false. As for the last constraint, it answers *maybe*: we have $[-2, 2] = 0$, the only possible deduction is that there maybe are values of v_1 and v_2 for which the constraint is satisfied.

2.2.1.3. *Solutions, Approximations*

Given a problem, we try to solve it, that is to find solutions to this problem. In the case of discrete variables, a solution is an instantiation of all variables, which corresponds to have a single value in each domain. In the case of continuous variables, it is considered as a solution a box containing only solutions or small enough. The fact that a solution is contained in small solution boxes is unclear. By being small, it reduces the chances of not containing any solution but does not delete them.

Solving a Constraint Programming problem is equivalent to compute the solution set or an approximation of it. In the case of discrete variables, list all the solutions is possible but can be expensive. Depending on the application, when a problem has a large number of solutions, we do not necessarily want to list them all. For example, consider $x \in \mathbb{Z}$ an integer variable, the constraints $x \geq 0, x \leq 999$ have a very large number of solutions (1000), and one can only want one or a subset of solutions. For continuous variables, the set of solutions can be infinite and to list them is then impossible. It is, for example, impossible to enumerate the real numbers between 0 and 1. Moreover, for a finite set of solutions, it is unlikely that the actual solutions are computer representable. In this case, an approximation of the solution set is accepted.

Definition 2.2.5 (Approximation)**.** A *complete approximation* (resp. *sound approximation*), of the solution set is an union of domain sequences or products $D_1 \dots D_n$ such that $D_i \subseteq \hat{D}_i$ and $S \subseteq \bigcup(D_1 \times \dots \times D_n)$ (resp. $\bigcup(D_1 \times \dots \times D_n) \subseteq S$).

Soundness guarantees that we find only solutions, while completeness guarantees that no solution is lost. On discrete domains, constraint solvers are expected to be sound and complete, *i.e.,* compute the exact set of solutions. This is generally impossible on continuous domains given that the reals are not computer representable, and one usually withdraw either soundness (most of the time) or completeness. In the first case, the resolution returns boxes that may contain points that are not solutions [BEN 99], we speak of outer approximation or *over-approximation*. This approximation is the most common. For most problems one seeks to know all the answers, and more importantly one wants to be sure not to lose any.

In the second case, all the returned boxes contain only solutions [COL 99, CHR 06]. In this case, we speak of inner approximation or *under-approximation*. This type of approximation appears in problems where we want to be sure that all points are good solutions, for example when controlling a robot arm surgery [ABD 96], we want to ensure that the robot arm remains in the area of operation. Another example is the control of a camera [BEN 04]. In this problem we want to determine the movements that must be performed by a camera (travel, zoom, *etc.*) to achieve an animation of a scene and a plan type specified by the user.

Remark 2.2.1 – The notions of correctness and completeness are different in Abstract Interpretation and Constraint Programming. To avoid ambiguity, we use the term over-approximation for a CP-complete AI-sound approximation and under-approximation for an AI-complete CP-sound approximation.

To solve a problem, that is, to compute the set of solutions (or an approximation in the case of reals) of the CSP, the CP-complete solving methods alternate two phases: propagation and exploration.

2.2.2. *Propagation*

First, propagation algorithms try to reduce the domains based on the constraints. The domains values that cannot be part of a solution, are deleted form domains. These values are called inconsistent.

2.2.2.1. *Consistency for one constraint*

Given a constraint C and domains, the consistency deletes all the inconsistent values for the constraint C from the domains. Several version of consistency have been proposed, such as *generalized arc consistency*, also known as *hyper-arc consistency* or *domains consistency* [MAC 77b], *path-consistency* [MON 74], *k-consistency* [FRE 78, FRE 82] or even *bound consistency*, also called *interval consistency* [DEC 03, HEN 95, APT 03, SCH 05]. They differ according to the domains type and their "strength". The strength is evaluated by the number of inconsistent values deleted from the domains. The stronger a consistency is, the more expensive it is in term of computation time and memory. We will come back to this point later. Theses consistencies are based on the notion of support.

Definition 2.2.6 (Support)**.** Let $v_1 \dots v_n$ be variables on finite discrete domains $D_1 \dots D_n, D_i \subseteq \hat{D}_i$, and C a constraint. The value $x_i \in D_i$ has a *support* if and only if $\forall j \in [\![1, n]\!], j \neq i, \exists x_j \in D_j$ such that $C(x_1, \dots, x_n)$ is true.

The most usual consistencies are the following.

Definition 2.2.7 (Generalized Arc-Consistency)**.** Given variables $v_1 \dots v_n$ on finite discrete domains $D_1 \dots D_n, D_i \subseteq \hat{D}_i$, and C a constraint. The domains are called *generalized arc-consistent* (GAC) for C if and only if $\forall i \in [\![1, n]\!], \forall x \in D_i$, x has a support.

The Generalized Arc-Consistency only keeps the values for which there is a solution for the constraint C, namely, values having a support.

Remark 2.2.2 – This consistency is also known as hyper-arc consistency or domains consistency [MAC 77b].

Example 2.2.4 – Let (v_1, v_2) be two variables on discrete domains $D_1 = D_2 = \{-1, 0, 1, 2, 3, 4\}$ and $v_1 = 2v_2 + 2$ be a constraint. The arc-consistent domains for this CSP are $D_1 = \{0, 2, 4\}$ and $D_2 = \{-1, 0, 1\}$.

Values -1 and 1 have been removed from v_1 domain because there exists no value for v_2 such that the constraints is satisfied. Similarly, if $v_2 \geq 2$ we can deduce that $v_1 \geq 6$. However the maximum value for v_1 is 4. Thus, the values greater than or equal to 2 can be removed from the domain of v_2.

The generalized arc-consistency for a binary constraint C has a time complexity in the worst case of $O(d^2)$ [MOH 86], with d the size of the largest domain. Note that this complexity does not take into account the complexity of the constraint. For each variable, every values must be tested in order to ensure that they have a support. For constraints with more than two variables, verifying that the domains are arc-consistent is NP-complete, and even when the constraints are linear [CHO 06]. However, there are specific algorithms for certain types of constraints. For instance there exists a dedicated algorithm for different versions of the $\mathrm{alldifferent}$ global constraint [SEL 02, HOE 04, GEN 08].

Other consistencies on discrete variables have been defined, such as the bound consistency based on the integer box representation. It is defined as follows.

Definition 2.2.8 (Bound Consistency)**.** Let $v_1 \dots v_n$ be variables on finite discrete domains $D_1 \dots D_n, D_i \subseteq \hat{D}_i$, and C a constraint. The domains are said *bound-consistent* (BC) for C if and only if $\forall i \in [\![1, n]\!]$, D_i is and integer interval, $D_i = [\![a_i, b_i]\!]$, and a_i and b_i have a support.

This consistency is weaker than the generalized arc-consistency, in fact it only checks whether the bounds of the domains have a support.

Remark 2.2.3 – There exist many different definition of the bound-consistency [CHO 06] (bound(D)-consistency, bound(Z)-consistency, bound(R)-consistency or even range consistency). The one we define above corresponds to the bound(Z)-consistency.

Example 2.2.5 – Let (v_1, v_2) be two variables on discrete domains $D_1 = D_2 = [\![-1, 4]\!]$ and $v_1 = 2v_2 + 2$ be a constraint. The bound-consistent domains for this CSP are $D_1 = [\![0, 4]\!]$ and $D_2 = [\![-1, 1]\!]$.

The bound-consistent and arc-consistent domains D_2 for the variable v_2 are the same (-1, 0, and 1) although the representations are different (integer interval or set). On the contrary the bound-consistent domain $D_1 = [\![0, 4]\!]$ for the variable v_1 is a more compact representation than the arc-consistent domain $D_1 = \{0, 2, 4\}$ but contains more values for v_1.

By intuition, one can say that the complexity of computing the bound consistency for a binary constraint is in the worst case $O(d^2)$ with d the maximum domains size. Indeed, in the worst case, the bound consistency keeps only one value in the domains. Thus it has to check if each value has a support. Deciding bound consistency of a constraint is NP-hard [SCH 05, CHO 06]. Note that there exist several studies giving effective algorithms to compute the bound consistency of specific constraints, such as the global constraint $\mathrm{alldifferent}$ [PUG 98, MEH 00, LÓP 03], the global constraint gcc [KAT 03, QUI 03] or set constraints [HEN 08].

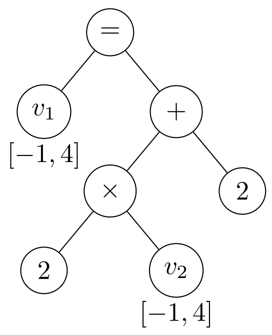
There are also different consistencies for continuous variables domains, such as *Hull-consistency*, *Box-consistency* or the extension of the generalized arc consistency for continuous constraints [BEN 94]. We only present here the Hull-consistency as defined in [BEN 97b].

Definition 2.2.9 (Hull-Consistency)**.** Let $v_1 \dots v_n$ be variables on continuous domains represented with intervals $D_1 \dots D_n \in \mathbb{I}, D_i \subseteq \hat{D}_i$, and C a constraint. Domains $D_1 \dots D_n$ are said *Hull-consistent* (HC) for C if and only if $D_1 \times \cdots \times D_n$ is the smallest box with floating-point bounds containing the solutions for C in $D_1 \times \cdots \times D_n$.

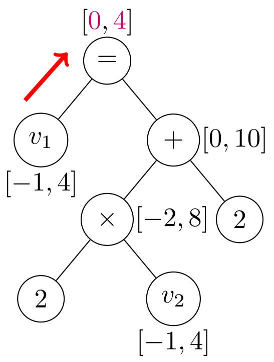
Example 2.2.6 – Consider two variables (v_1, v_2) on continuous domains $D_1 = D_2 = [-1, 4]$ and the constraint $v_1 = 2v_2 + 2$. The hull-consistent domains for this CSP are $D_1 = [0, 4]$ and $D_2 = [-1, 1]$.

There exist several algorithms computing the Hull-consistency for a constraint. In 1999, Benhamou *et al.* in [BEN 99] propose and algorithm HC4-Revise based on the a tree representation of the constraint in order to compute the Hull-consistency in a linear time $O(e)$ with e the number of unary or binary operators in the constraint. More recently a new algorithm Mohc-Revise [ARA 10] studies the monotony of the constraint so as to compute the Hull-consistency.

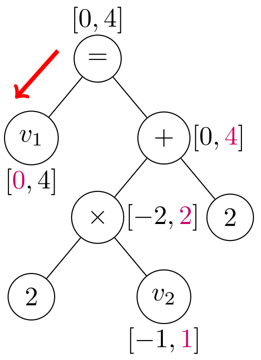
Example 2.2.7 – Let us consider the CSP from Example 2.2.6. The constraint $v_1 = 2v_2 + 2$ is represented with the following syntax tree:



The nodes corresponding to a variable are tagged with the corresponding variable domain. The HC4-Revise algorithm performs a first pass in tree from the leaves to the root. During this first pass, it uses the interval arithmetic [MOO 66] in order to tag each node by its interval of possible values. For instance, the node $(\times)$ is tagged $2 \times [-1, 4] = [-2, 8]$. The root tag is computed using its children but also given the type of constraint. In this example, the constraint being an equality the root tag is equal to the intersection of its children nodes interval.



Then a second pass is performed, but this time from the root to the leaves. It gives to all the nodes the information obtained at the root and can modify the possible values at each node. The node $(\times)$ is now tagged $[0,4]-2=[-2,2]$.



By applying this algorithm, we obtain the same results as in the example 2.2.6 $(D_1=[0,4], D_2=[-1,1])$.

Remark 2.2.4 – Note that it is not always the case, as the HC4-Revise algorithm can computes an approximation of the Hull-consistency. For instance when there exists multiple occurrences of a variable in the constraint, the result obtained with the HC4-Revise algorithm can be an over-approximation of the hull-consistent domains.

Figure 2.5 shows the different consistencies obtained in examples 2.2.4, 2.2.5 and 2.2.6. In Figures 2.5(a) and 2.5(b) the points correspond to the Cartesian product of the consistent domains. There are 3 solution points and are represented by green

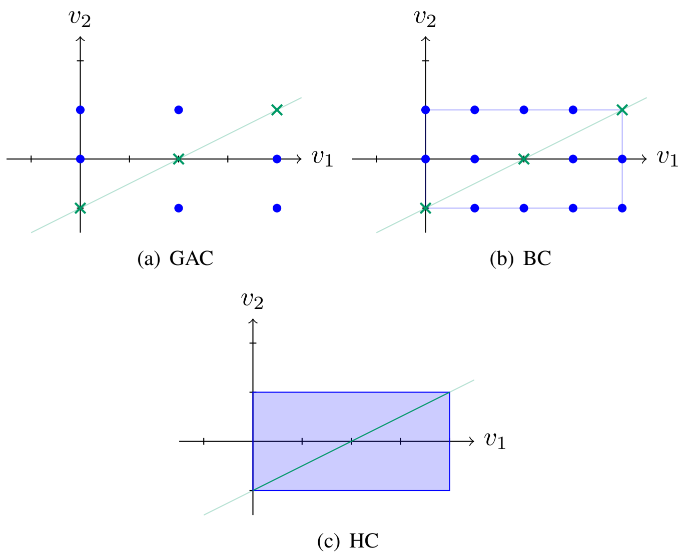


Figure 2.5. *Différentes consistances pour la contrainte* $v_1 = 2(v_2 + 1)$.

squares. With the Cartesian product of the arc-consistent domains (Figure 2.5(a)), we obtain 9 points, while with the bound-consistent domains (Figure 2.5(b)), we obtain 15 points. We can deduce that, the stronger the chosen consistency is, the higher the ratio of number of solution points on the total number of points is. Figure 2.5(c) shows the hull-consistent box. The solutions are also in green and correspond to the diagonal in the box. We can see that in this box there are more non-solution points than solution points.

For a given constraint C, an algorithm taking as input the domains and removing from them the inconsistent values is called a *propagator* and is denoted by ρ_C. The HC4-Revise algorithm is a propagator for the Hull-consistency. The propagators usually return and over-approximation of S. Sometimes they compute exactly the consistent domains, but as this can be very expensive or intractable, they generally return an over-approximation of the consistent domains.

For several constraints, the propagators of each constraint are iteratively applied until a fixpoint is reached. This is called the propagation loop.

2.2.2.2. *Propagation Loop*

As the domains representation, previously introduced (Definitions 2.2.2, 2.2.3, 2.2.4), form finite complete lattices for inclusion, it is sufficient to compute the consistency for each constraint until the fixpoint is reached. It has been demonstrated by Apt in 1999 in [APT 99] or by Benhamou in 1996 in [BEN 96] that the order in which the consistences are applied is irrelevant. Indeed, as the lattices are complete, any subset has a unique least element: the consistent fixpoint.

However the order in which the consistencies, and therefore the propagators, are applied influences the speed of convergence, that is the number of iterations required to reach the fixpoint. Different propagation strategies are possible. The naive strategy is to execute all the constraints propagator at each iteration. Another one is to execute only the propagator of the constraints containing at least one variable that has been modified in the previous iteration. This strategy is generally used. Finding an efficient algorithm computing the consistent domains for a conjunction of constraints in the general case is a real scientific challenge. In the following, the term "complexity" is used to refer to the worst case complexity.

Since the 1970s, many algorithms have been developed for the generalized arc consistency for binary constraints (AC) or for any constraints (GAC), each one improving the complexity of the previous algorithm. The first algorithm AC1 described by Mackworth in 1977 [MAC 77a] computes the arc-consistent domains for binary constraints. It verifies for each constraint (p), for each variable (n), and for each possible value (at most d) if it has a support (d^2). This algorithm has a time complexity of $O(d^3np)$ where d is the size of the largest domain, n is the number of variables and p the number of constraints. This algorithm is rather basic and an improved version, called AC3 is given in the same paper [MAC 77a]. AC3 uses a list in order to only propagate the constraints in which at least one variable has been modified. AC3 has a time complexity of $O(pd^3)$ and a memory complexity of $O(p)$. The same year, an extended version of AC3 no longer restricted to binary constraint is presented, GAC3 [MAC 77b]. This algorithm has a time complexity of $O(pk^3d^{k+1})$ and a memory complexity of $O(pk)$, with k the maximum constraint arity, which is the maximum number of variables that can be in a constraint.

The algorithms AC4, proposed by Mohr and Henderson in 1986 [MOH 86], and GAC4, proposed by Mohr and Masini in 1988 [MOH 88], have a worst case time complexity which is optimal [BES 06]. AC4 has a time and memory complexity of $O(pd^2)$. GAC4 has a time complexity of $O(pkd^k)$.

Since then, several algorithms have been designed: AC5 [HEN 92], AC6 [BES 94], AC7 [BES 99], AC2001 [BES 01], AC3.2 [LEC 03] to name few, ingenuously competing to reduce in practice the memory complexity and particularly the computation time. A comparison and the complexity of some of these algorithms as well as those given above can be found in [MAC 85] and [BES 06].

While there have been many papers and algorithms for the efficient computation of generalized arc consistency, the bound consistency has been the subject of less studies. Moreover, there exist several different definitions of the bound consistency [CHO 06] (bound(D)-consistency, bound(R)-consistency, range consistency) in addition to the one considered in this book which has changed names several times over the years.

In the continuous variables case, the HC3 algorithm proposed in 1995 [BEN 95] computes an approximation of the hull consistency for a conjunction of constraints. Then in 1999 [BEN 99], the HC4 algorithm is proposed. The propagation loop applies the HC4-Revise algorithm seen earlier for each constraint containing at least one variable that has been modified. More recently, the Mohc algorithm studying the monotony of the constraints to propagate has been developed [ARA 10, ARA 12].

2.2.3. *Exploration*

Generally, the propagation is not sufficient to find the solutions. During the second step of the resolution process, assumptions about the variables values are made. In the case of integer variables, values are given to variables iteratively until a success (a solution is found) or a fail (a constraint is false, an empty domain) is obtained. For real variables, the idea is the same except that the domains are split into two smaller sub-domains. This exploration mechanism introduces the notion of *choice point* in the resolution process.

2.2.4. *Resolution Scheme*

The solving process alternates between phases of reduction and choice. At any point of the resolution, the propagation is first performed to remove as soon as possible inconsistent values from the domains and avoid unnecessary computations. Thus, consistency is maintained throughout the resolution. Three cases are possible:

1) a solution is found, it is stored;

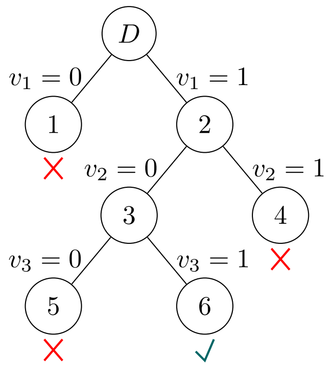
2) no solution is possible, in other word, at least one domain is empty or a constraint is false;

3) or none of the above.

In the first two cases, the process returns to a previous choice point to make a new hypothesis. Several cases are possible, return to the last choice point, called *backtracking*, or in the case of a failure, returning to the most likely point of choice responsible for the failure, in this case it is called *backjumping* [DEC 90]. Several techniques are used to determine the node to go back to, it can be based on the dependencies between the variables or use learning during the solving process. In any case, if the search space is not yet fully explored, a choice is made, a variable and a value are chosen in the discrete case or a domain in the continuous case. This choice is taken into account by instantiating the variable or cutting the chosen domain, then back to the first step: the propagation is performed in this new point of resolution. Thus, consistency is maintained throughout the resolution.

This solving mechanism correspond to a search tree in which the nodes are the choice points, and the arcs correspond to a variable instantiation or a domain split.

Example 2.2.8 – Consider a CSP on boolean variables v_1, v_2 and v_3 on domains $D_1 = D_2 = D_3 = \{0, 1\}$ and with the constraint: $v_1 \text{ and not } v_2 \text{ and } v_3$. We have the following search tree:



where $D = D_1 \times D_2 \times D_3$. Starting from the initial search space, a first choice is made: $v_1 = 0$. After this instantiation (node 1) the constraint is false. The exploration is stopped and goes back to the root where a new choice is made: $v_1 = 1$ (node 2). The exploration continues until all the nodes are visited. In this example, a red cross corresponds to a failure and a green tick to a solution.

The general scheme of a discrete solver using the generalized arc consistency is exemplified Algorithm 2.1. Note that the branching can also be binary ($v_1 = a$ on one branch, and $v_1 \neq a$ on the other with a a value in D_1). Similarly, a continuous solver is given Algorithm 2.2. It uses the hull-consistency. In these examples, the resolution stops when:

– all solutions are found, that is when:
  - all solutions have been listed in the discrete case;
  - all computed boxes only contain solutions or are smaller than a certain accuracy in the continuous case.

– it has been proven that there is no solution, in other words, the search space has been fully explored without success.

Note that in both the discrete and the continuous case, the resolution process can be modified in order to stop as soon as the first solution is found.

```
int $j \leftarrow 0$                          /* $j$ indicates the depth in the search tree */
int op[]      /* at depth $j$, stores the index of the variable on which the hypothesis is made,
uniformly initialized at $0$ */
int width[] /* at depth $j$, stores the width of the tree already explored, uniformly initialized
at $0$ */
integer Cartesian product $e \in \mathbb{S}$
list of integer Cartesian product sols                              /* stores the solutions */
$e \leftarrow D$                                   /* initialization with the initial domains */
repeat
  $e \leftarrow$ generalized arc-consistency($e$)
  width[$j$]++
  if $e$ is a solution then
    /* success */
    sols $\leftarrow$ sols $\cup\, e$
  end if
  if $e = \emptyset$ or $e$ is a solution then
    /* back to the last backtrackable point */
    while $j \geq 0$ et width$[j] \geq |D_{op[j]}|$ do
      width$[j] \leftarrow 0$
      $j--$
    end while
  else
    /* new hypothesis */
    choose a variable $v_i$ to instantiate
    op$[j] \leftarrow i$
    $j++$
  end if
  if $j \geq 0$ then
    assign $v_{op[j-1]}$ to the (width$[j]$ + 1)-th possible value   /* backtrackable point */
  end if
until $j < 0$
```

Algorithm 2.1: A discret solver using integer Cartesian product to represent the domains.

In both solving methods, the selection criteria of the variable to instantiate or domain to cut is not explicitly given. This is because there is no unique way to choose the domain to be cut or the variable to instantiate and it often depends on the problem to solve. The next section describes some of the choice strategies or existing exploration strategies.

```
list of boxes sols ← $\emptyset$                                  /* stores the solutions */
queue of boxes toExplore ← $\emptyset$                        /* stores the boxes to explore */
box $b$ ← $D$                                        /* initialization with the initial domains */
push $b$ in toExplore
while toExplore $\neq \emptyset$ do
  $b$ ← pop(toExplore)
  $b$ ← Hull-consistency($b$)
  if $b \neq \emptyset$ then
    if $b$ only contains solutions or $b$ small enough then
      /* success */
      sols ← sols $\cup$ $b$
    else
      /* new hypothesis */
      cut $b$ in $b_1$ and $b_2$ by splitting along one of its dimensions
      push $b_1$ and $b_2$ in toExplore
    end if
  end if
end while
```

Algorithm 2.2: A classical continuous solver.

2.2.5. *Exploration Strategies*

Several choice strategies have been designed to determine the order in which the variables should be instantiated or the domains should be split. Depending on the choices made, performances may be very different. In the discrete case, the best known is maybe the one presented in 1979 by Haralick and Elliott [HAR 79] called *first-fail*. It chooses the variable with the smallest domain. The idea is the following: the earlier one fails, the bigger is the subtree cut in the search tree.

"To succeed, try first where you are most likely to fail"
– Robert M. Haralick et Gordon L. Elliott [HAR 79].

Figure 2.6 shows that the earlier the failure appears in the search tree, then the bigger is the subtree cut from the search tree. In Figure 2.6(a) failure occurs later and only a small subtree in the search tree is cut. However in Figure 2.6(b) failure occurs earlier and a larger subtree in the search tree is cut.

Figure 2.7 compares the search tree obtained with the strategy *first-fail* (Figure 2.7(b)) to the one obtained with the strategy instantiating variables with large domains first. Note that these two trees have the same area. In case of failure, the cut in the search tree is greater when the strategy *first-fail* is used.

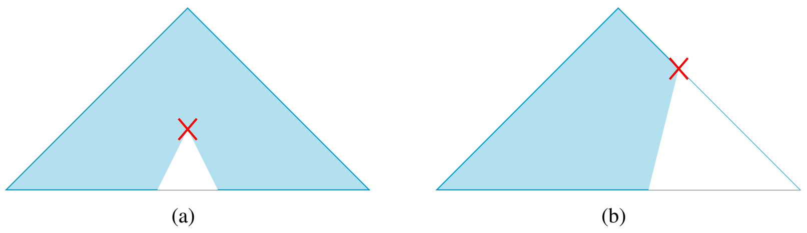


Figure 2.6. *Comparison of the subtree cut in the search tree given the node where the failure appears.*

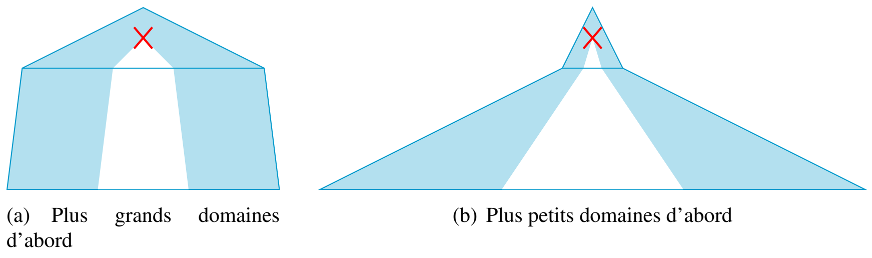


Figure 2.7. *Comparaison de la stratégie qui consisterait à instancier les variables de grands domaines en premier (2.7(a)) à la stratégie* first-fail *(2.7(b)).*

Following this idea, the heuristic proposed in [BRÉ 79] chooses the variable with the smallest domain (dom) and appearing in the biggest number of constraints deg. In other words, the chosen variable is the one maximizing $\mathrm{dom} + \mathrm{deg}$. Then in 1996 another strategy is presented [BES 96] choosing the variable maximizing the ratio $\mathrm{dom}/\mathrm{deg}$. In 2004 [BOU 04], a heuristic choosing the variable maximizing the ratio $\mathrm{dom}/w\mathrm{deg}$, with w a weight associated to each constraint, is introduced. The weights are uniformly initialized to 1 at the beginning of the resolution. Then, each time a constraint is the reason of a failure, its weight is augmented. The idea is the following: instantiate in priority variables appearing in constraints difficult to satisfy. The earlier this constraint generates a failure, the bigger is the subtree cut from the search tree. Other heuristics as a survey is given in [GEE 92, GEN 96, BEE 06].

Once the variable to instantiate is chosen, one needs to choose to which value it should be instantiated. Here too, many strategies have been developed, choosing the value maximizing the number of possible solutions[DEC 87, KAS 04], the product of the domains size (*promise*) [GIN 90], the sum of the domains size (*min-conflicts*) [FRO 95].

In the case of continuous variables, a domain is generally cut in half. Several heuristics to split a domain are available, *largest-first* [RAT 94] choosing the largest domain, thus the domains size is rapidly reduced; *round-robin* splitting the domains one after the other, to ensure that all the domains are cut; or even *Max-smear* [HAN 92, KEA 96] choosing to split the domain maximizing the *smear* function of the constraints Jacobian matrix (containing the partial derivatives of each constraint). This strategy corresponds to splitting the domain of the variable with the biggest slope in the constraints.

2.2.6. ***Discrete/Continuous Comparison***

It is important to note that the resolution scheme is significantly different depending on the type of variable (integer or real). In practice, the solvers are dedicated to a particular type of variable. There are many discrete solvers. Among the most known are GeCode [SCH 02], GnuProlog with constraints [DIA 12], Jacop [WOL 04], Comet [HEN 05], Eclipse [APT 07] and Minion [GEN 06]. There are much less continuous solvers in Constraint Programming. The best known are Declic [BEN 97a], Numerica [HEN 97], RealPaver [GRA 06] and Ibex [CHA 09a]. Prolog IV [COL 94] and Choco 3.0 [PRU 14, FAG 14] incorporates both a discrete and a continuous solver.

Implementation tricks are needed to solve problems containing both integer variables and real variables. In the following, we use the terms "mixed problem" or "continuous-discrete problem" for this type of problems. If the selected solver is discrete, real variables are discretized, the possible values for the variables are listed for a given step.

Example 2.2.9 – Let x be a real variable which value is between 0 and 0.5, with a step of 0.1. Once discretized, its domain is $\{0, 0.1, 0.2, 0.3, 0.4, 0.5\}$.

This method offers the possibility to treat mixed problems but strongly depends on the chosen step. Indeed, if the chosen step is large, solutions could be lost but the variables domain is small and can easily be stored. On the contrary, if the step is too small, there are less chances of losing solutions but the domains size is very large and it explodes the combinatory of the problem. This method is used among others in the discret solver Choco 2.0 [TEA 10]. In this case, the solver is no longer CP-complete, it no longer guarantees that no solution is lost. But it is still CP-sound, if it returns an assignment, it is a solution.

If the selected solver is continuous, integrity constraints can be added to the problem. These new constraints specify which variables are integer and thus allows the solver to refine the bounds of these variables domain. During the propagation of these constraints, the domains bounds are rounded to the nearest integer in the proper direction. This method requires the bound consistency for the discrete variables and does not allow the generalized arc consistency to be used. This method has been developed in the continuous solver RealPaver [GRA 06].

Another alternative is to add discrete global constraints to a continuous solver [BER 09] or to create mixed global constraints [CHA 09b] to treat within the same constraint both continuous and discrete variables. Thus, each variable benefit from a suitable constraint for its type (discrete or continuous). However, this method depends on the problem and demands for each problem the necessary global constraints as well as an *ad hoc* consistency.

2.2.7. *Conclusion*

Constraint Programming can efficiently solve constraint satisfaction problems. While solving methods do not depend on the problem at hand, they are highly dedicated to the type of variables (discrete or continuous) of the problem. This specialization, although restrictive, makes solving methods effective. Many heuristics have been developed and used during the exploration phase to improve the results obtained in terms of both the quality of the solutions and the computation time. However, one major obstacle to the development of Constraint Programming is the lack of tools and methods to solve mixed problems (with both discrete and continuous variables). For instance, there is no representation for mixing in a single CSP integers and reals while maintaining their types.

2.3. Synthesis

In the previous sections, we have introduced some concepts of Abstract Interpretation and Constraint Programming, two areas of research that seems very distant. Yet there are similarities both in the underlying theory and in the techniques used. This section highlights some of these links and some significant differences.

2.3.1. *Links between Abstract Interpretation and Constraint Programming*

There are links between Constraint Programming and Abstract Interpretation, for instance, both research areas rely on the same theoretical framework (lattices and fixpoint). But if you look closer there are also notable differences. We present here some of the similarities and differences between these two research areas.

As mentioned previously, Constraint Programming and Abstract Interpretation rely on the same theoretical framework, and more importantly, the main goal is the same: compute an over-approximation of a desired set, expressed as solutions of constraints in Constraint Programming and as the properties of a program in Abstract

Interpretation. In the particular case of discrete CSP, the approximation should be equal to the desired set. Consistency and propagation can be seen as a narrowing on abstract domains, because they both help reduce the domains while remaining above the fixpoint. In Abstract Interpretation, the narrowing operator can either be used after a widening, which ensures that the abstract domains stay larger than the desired fixpoint, or in the particular case of local iterations. In Constraint Programming, domains are always reduced and the propagation is used to reduce the domains as soon as possible. Note that in both cases, it is allowed not to reach the fixpoint. For example, in Constraint Programming for certain constraints, such as the global constraint `nvalue`, computing the consistency is NP-hard [BES 04] and it is allowed to compute an over-approximation of the consistent domains even for discrete constraints.

Remark 2.3.1 – On the theoretical framework, another significant difference between Constraint Programming and Abstract Interpretation should be noted. In Abstract Interpretation, fixpoints theorems are fundamental and the basis of loops analysis, a key step in the analysis of a program. In Abstract Interpretation, the lattices are usually infinite, while in Constraint Programming, lattices used for consistencies are always finite. They are built from sets starting bounded (initial domains) and their components must be computer representable (integers or floating point), so in a finite number.

These two research areas rely on the computation of fixpoint or of over-approximations of fixpoints. However, the techniques used to compute them are significantly different. First, in Constraint Programming, while solving a given problem, for a series of choice point, the approximations are strictly decreasing, except when the fixpoint is reached. While in Abstract Interpretation the approximations can increase, for instance when approximating a loop. Secondly, Constraint Programming aims at completeness by improving solutions using refinement, while Abstract Interpretation generally embraces incompleteness.

Furthermore, the static analyser precision is conditioned by the chosen abstract domain and the used operators. When a refinement is performed, domains become more precise, which can be done by either restarting the analyzer with the obtained domains [CLA 03], either manually, which requires changing the analyzer [BER 10]. Another way to improve the accuracy is to add local iterations [GRA 92]. To sum up, the accuracy of an abstract domain is not defined as such it is implied by the choice made for the analysis (abstract domains, transfert function, widening, narrowing...) In contrast, Constraint Programming integrates for continuous domains an explicit definition of accuracy. The choice of abstract domain in a Constraint Programming solver does not change its precision (which is fixed), but its efficiency (the amount of computation needed to achieve the desired accuracy).

Another significant difference is that all the techniques in Abstract Interpretation are parametrized by abstract domains. On the contrary, Constraint Programming solvers are highly dependent on the variables type (integer or real) and are especially dedicated to one domain representation. There exists no Constraint Programming solver for which the domains representation is a parameter, in particular, there is no mixed solvers. Section 2.2, we saw that in Constraint Programming there is only three representations for the variables domain given the type of the variable. On the contrary, in Abstract Interpretation, an important research effort has been made on abstract domains. This has lead to the development of many Cartesian or relational domains such as polyhedra, octagons, ellipsoid, to name a few. Note that they can depend on the variable type, like the binary decision diagrams for binary variables [BER 10], but more importantly, they can analyse different variables type within the same analyzer.

2.3.2. *Analysis*

Constraint Programming allows one to model a wide variety of problems under a same and unique form, the CSP. It offers a wide variety of constraints, including, in the case of discrete variables, more than three hundreds global constraints. They can be used to describe complex relations between the variables and come with efficient propagation algorithms. Number of studies have focused on improving the complexity and therefore the efficiency of these algorithms. Another strong point of Constraint Programming is that it provides generic solving methods that do not depend on the problem to solve, and a single constraint can be used in different problems. In order to improve these solving methods performances, many heuristic have been developed, such as the variable or value choice heuristic for the instantiation, or the choice heuristic for the domain to split. However, Constraint Programming offers very few representations for the domains, and the solving methods are restricted to only one type of variables (discrete or continuous).

On its side, Abstract Interpretation provides a large number of representations, abstract domains. These can be of different shapes (box, polyhedron, ellipse, . . .) and are not defined for a particular type of variables. Thus, they may represent both discrete and continuous variables. Moreover, using several abstract domains allows the analyzers to effectively deal with very large programs, containing a large number of variables and lines of code. Another advantage of abstract domains is that they come with efficient algorithms for the widening and narrowing operators to compute in a small number of iterations an approximation of the fixpoint. However this approximation is not always very precise, even after applying the narrowing operator or local iterations.

To sum up, Constraint Programming offers very few representations for the domains but efficient algorithms, while in Abstract Interpretation there are very rich representations for the domains and algorithms to compute over-approximations but no solving algorithm (in the sense that the accuracy is not easily configurable).

There have already been works at the frontier between Abstract Interpretation and Constraint Programming. For instance, Constraint Programming has been used to verify programs [COL 07], to analyse characteristic models and automatically generate configuration tests [HER 11b], to verify constraint satisfaction problems model [LAZ 12], or even to improve the results obtained by a static analyzer [PON 11].

Another approach to program verification is to use a satisfiability solver [KIN 69]. Recent years have seen significant improvements in the methods of Boolean satisfiability (SAT) and non-Boolean satisfiability (SMT) [KRO 08], as well as in their applications (*e.g.,*, the model checker CBMC [KRO 14]). Furthermore, D'Silva *et al.,* [D'S 12] have recently proposed to express the SAT algorithms in the context of Abstract Interpretation (using fixpoints and abstractions), a promising way to cross-pollination between these two research areas.

The work presented in this book is similar, except that we are at the intersection of Abstract Interpretation and Constraint Programming. There are of course similarities between the solving process in Constraint Programming and in SAT/SMT, however both the model and the solving methods differ. Resolution methods in SAT and SMT used a model based on Boolean variables, for which the algorithms are dedicated. Constraint Programming combines the constraints on any type of variables and sometimes loses efficiency to gain expressiveness.

Chapter 3

Abstract Interpretation for the constraints

In this chapter we give unified definitions for the different components of the resolution process in Constraint Programming. With these new definitions an unique resolution method can be defined for any abstract domain. This new resolution scheme does not depend on the variables type anymore. Moreover, the definition of the abstract domains in Constraint Programming gives the possibility to solve problems using non-Cartesian domains representations.

3.1. Introduction

In Constraint Programming, solving techniques strongly depend on the variables type, and even are dedicated to a type of variables (integer or real). If a problem contain both integer and real variables, there exist no more solving methods. Three solutions are possible to solve this kind of problem with Constraint Programming: the integers are transformed into reals and integrity constraints are added to the problem (the propagation of these new constraints refine the bound of the integer variables [GRA 06]); the reals are discretized, the possible values for the real variables are enumerated with a given step [TEA 10]; or a discrete and a continuous solver are combined [COL 94, FAG 14]

By looking more closely, we can see that regardless of the resolution method used, it alternates propagation and exploration phases. We can even go further and say that these two methods differ only on three points: the representation of the domains, the consistency used and the way domains are cut. The splitting operator used during the solving process strongly depend on the chosen representation. Indeed, one needs to know some characteristics of the chosen representation, such as the size to cut it in smaller elements. Similarly, the consistency used strongly depend on the chosen representation. Indeed, the generalized arc consistency would not be used if the domains are represented using integer intervals. If domains are represented using integer intervals, the bound consistency is used. If the integer Cartesian product is used, then the generalized arc consistency is more appropriate. And if the domains are represented using floating point intervals, the hull-consistency is more suitable. Finally the consistency does not depends only on the domains representation but also on the constraints.

Inspired by abstract domains in Abstract Interpretation, we define abstract domains Constraint Programming, as an object that contains, among other things, a computer representation, a splitting operator and a function to compute its size. The consistency could be added to the abstract domains, but we prefer to dissociate it since it does not depend only on the domains representation. With this definition of abstract domains in Constraint Programming, we obtain an unique solving method that no longer depend on the variables type, or the domains representation. In this new solving method, the abstract domain is a parameter. In addition, we are no longer restricted to existing Cartesian representations, but can define new representation in the same way that in Abstract Interpretation.

3.2. Unified Components

First, we define all the necessary bricks for the development of an unique solving method. Namely, the consistency, the splitting operator and of course the abstract domains for Constraint Programming. These definitions are based on notions of order, lattices and fixpoint.

3.2.1. *Consistency and Fixpoint*

Given a partially ordered set with the inclusion and a constraint, we can define the consistent element as the least element of the partially ordered set if it exists. Similarly for a conjunction of constraints, the set of least elements for each constraint forms a partially ordered set with the inclusion and the consistent element is therefore defined as the least element of this set. From this observation derived the following definitions and propositions.

Let E be a subset of $\mathcal{P}(\hat{D})$, where $\hat{D}$ is the initial search space, with the inclusion as a partial order. This set corresponds to the chosen representation for the domains. We will write E^f for $E\backslash\emptyset$. In the following, we will restrain the definitions to cases where E is closed by intersection, as this is sufficient for classical cases. Note that if E is closed by intersection then it possess a greatest lower bound and is a directed lattice.

Definition 3.2.1 (E-Consistency)**.** Consider a constraint C. An element e is E-consistent for C if and only if it is a least element of E containing all the solutions for C, S_C. In other words, it is the least element of

$$\mathbb{C}_C^E = \{e' \in E, S_C \subseteq e'\}$$

All the main consistencies defined section 2.2.2, whether discrete or continuous, are included in this definition. The following propositions combine each existing consistency to its corresponding subset. These propositions ensure that the main consistencies are properly included in Definition 3.2.1.

Proposition 3.2.1. *Let $\mathbb{S}$ be the set of Cartesian products of finite subsets of integers. The $\mathbb{S}$-consistency is the generalized-arc-consistency (GAC) (Definition 2.2.7).*

Proof.

GAC $\Longrightarrow$ $\mathbb{S}$-consistent.

We first prove that GAC $\Longrightarrow$ $\mathbb{S}$-consistent. Let $D = D_1 \times ... \times D_n$ GAC for a constraint C. D obviously contains all the solutions, and we need to prove that it is the smallest such element of $\mathbb{S}$. Let $D' \in \mathbb{S}$ strictly smaller than D. Then there is a i such that $D'_i \subset D_i$ and a $v \in D_i \setminus D'_i$. Since $v \in D_i$ and D is GAC, there also exist $x_j \in D_j$ for $j \neq i$ such that $(x_1 \ldots v \ldots x_n)$ is a solution. This solution is not in D' and thus any $D' \subset D$ loses solutions. $\diamond$

$\mathbb{S}$-consistent $\Longrightarrow$ GAC.

We now prove that $\mathbb{S}$-consistent $\Longrightarrow$ GAC. Let $D = D_1 \times \cdots \times D_n$ $\mathbb{S}$-consistent for a constraint C. Let $i \in [\![1, n]\!]$ and $v \in D_i$. Suppose that for all other $x_j \in D_j$, $(x_1 \ldots v \ldots x_n)$ is not a solution. We can construct the set $D_1 \times \cdots \times (D_i \setminus \{v\}) \times \cdots \times D_n$ strictly smaller that D containing all the solutions. Hence D is not the smallest element. There can be no such i and v, and D is GAC. $\diamond$

A domain is GAC if and only if it is $\mathbb{S}$-consistent.

□

Proposition 3.2.2. *Let $\mathbb{IB}$ the set of integer boxes (Cartesian product of finite interval of integers). The $\mathbb{IB}$-consistency is the bound-consistency (BC) (Definition 2.2.8).*

Proof.

BC $\Longrightarrow$ $\mathbb{IB}$-consistent.

We first prove that BC $\Longrightarrow$ $\mathbb{IB}$-consistent. Let $D = D_1 \times \cdots \times D_n$ BC for a constraint C. D obviously contains all the solutions, and we need to prove that it is the smallest such element of $\mathbb{IB}$. Let $D' \in \mathbb{IB}$ strictly smaller than D. Then there is a i such that $D'_i \subset D_i$. Let v be one of the bound of D_i such that $v \notin D'_i$. Since D is BC, there also exists $x_j \in D_j$ for $j \neq i$ such that $(x_1 \dots v \dots x_n)$ is a solution. This solution is not in D' and thus any $D' \subset D$ loses solutions. $\diamond$

$\mathbb{IB}$-consistent $\Longrightarrow$ BC.

We now prove that $\mathbb{IB}$-consistent $\Longrightarrow$ BC. Let $D = D_1 \times \cdots \times D_n$ $\mathbb{IB}$-consistent for a constraint C. Let $D_i = [\![a_i, b_i]\!]$. Suppose that for all other $x_j \in D_j$, $(x_1 \dots a_i \dots x_n)$ is not a solution. We can construct the set $D_1 \times \cdots \times [\![a_i + 1, b_i]\!] \times \dots D_n$ strictly smaller that D containing all the solutions. Hence D is not the smallest element. There can be no such i and v, and D is BC. $\diamond$

A domain is BC if and only if it is $\mathbb{IB}$-consistent.

$\square$

Proposition 3.2.3. *Let $\mathbb{B}$ be the set of boxes. The $\mathbb{B}$-consistency is the Hull-consistency (HC) (Definition 2.2.9).*

Proof.

HC $\Longrightarrow$ $\mathbb{B}$-consistent.

We first prove that HC $\Longrightarrow$ $\mathbb{B}$-consistent. Let $D = D_1 \times \cdots \times D_n$ HC for a constraint C. D obviously contains all the solutions, and we need to prove that it is the smallest such element of $\mathbb{B}$. Let $D' \in \mathbb{B}$ strictly smaller than D. Then there is a i such that $D'_i \subset D_i$. Let $I \in D_i \setminus D'_i$. Since $I \in D_i$ and D is HC, there also exists $I_j \in D_j$ for $j \neq i$ such that $C(I_1, \dots, I, \dots, I_n)$. This solution is not in D' and thus any $D' \subset D$ loses solutions. $\diamond$

$\mathbb{B}$-consistent $\Longrightarrow$ HC.

We now prove that $\mathbb{B}$-consistent $\Longrightarrow$ HC. Let $I \in \mathbb{B}$, $\mathbb{B}$-consistent for a constraint C. Then I is the least element of $\mathbb{C}_C^{\mathbb{B}}$ which corresponds to the smallest Cartesian product of intervals with floating-point bounds containing all the solutions for C. Hence, by Definition 2.2.9, I it is HC. $\diamond$

A domain is HC if and only if it is $\mathbb{B}$-consistent.

$\square$

Definition 3.2.1 of E-consistency thus generalizes the existing consistency. For any $E \subseteq \mathcal{P}(\hat{D})$ closed by intersection, consistency is well-defined. From this definition of E-consistency derives the following proposition:

Proposition 3.2.4. *If E is closed under infinite intersection, $\mathbb{C}_C^E$ is a complete lattice and there exists a unique E-consistent element for C in E. If it exists, this element is written $\mathbf{C}_C^E$.*

Remark 3.2.1 – Note that if E is not closed under intersection, then the least element of $\mathbb{C}_C^E$ does not always exist. For instance, in the particular case where C is a circle and E the set of convex polyhedra, there exists no smallest polyhedron containing all the solutions. This is due to the fact that polyhedra are not closed by intersection.

Proof.

Let $\mathbf{C}_C^E = \bigcap\limits_{e \in E, S_C \subseteq e} e.$

Least elements.

We first prove that $\mathbf{C}_C^E$ is the least element of $\mathbb{C}_C^E$. Let $A \in \mathbb{C}_C^E$ strictly smaller than $\mathbf{C}_C^E$, we have $S_C \subseteq A$. Since $\mathbf{C}_C^E$ is the intersection of all the elements containing the solutions S_C, A cannot be smaller than $\mathbf{C}_C^E$. Thus $\mathbf{C}_C^E$ is the least element of $\mathbb{C}_C^E$. $\diamond$

Unicity.

We now prove that $\mathbf{C}_C^E$ is unique. Suppose that there is an element $B \in \mathbb{C}_C^E$ such that B is also a least element of $\mathbb{C}_C^E$. If such an element exists, then it also contains all the solutions, $S_C \subseteq B$. since $\mathbf{C}_C^E$ is the intersection of all the elements containing the solutions S_C, we have $\mathbf{C}_C^E \subseteq B$. Hence $\mathbf{C}_C^E$ is unique. $\diamond$

$\mathbf{C}_C^E$ is unique and the least element of $\mathbb{C}_C^E$.

□

Definition 3.2.1 can easily be extended for a conjunction of constraints as follows:

Definition 3.2.2 (E-consistency for a conjunction of constraints)**.** Let $e \in E$, e is E-consistent for $C_1 \dots C_p$ (or a subset) if and only if it is a least element for $\mathbb{C}_{C_1 \wedge \dots \wedge C_p}^E = \{e' \in E, S_{C_1 \wedge \dots \wedge C_p} \subseteq e'\}$. If such an element exists, it is written $\mathbf{C}_{C_1 \wedge \dots \wedge C_p}^E$. If there is no ambiguity $\mathbb{C}_{C_1 \wedge \dots \wedge C_p}^E$ is written $\mathbb{C}^E$ and its least element $\mathbf{C}^E$.

This definition will preferably be used in the case where E is closed under intersection.

Proposition 3.2.5. *If E is closed by intersection then $\mathbf{C}^E$ exists and is unique. In addition, the set of all $\mathbf{C}_{C_{i_1} \wedge \dots \wedge C_{i_k}}^E$ for $i_1 \dots i_k \in [\![1, p]\!]$ forms a lattice for inclusion and $\mathbf{C}_{C_1 \wedge \dots \wedge C_p}^E$ is its least element.*

Proof.

Unicity.

Unicity of $\mathbf{C}^E$ directly comes from Proposition 3.2.4. $\diamond$

Complete Lattice.

We first prove that the set of all $\mathbf{C}^E_{C_{i_1}\wedge\dots\wedge C_{i_k}}$ for $i_1 \dots i_k \in [\![1,p]\!]$ forms a lattice for inclusion. Let $\mathbf{C}^E_{C_i}$ and $\mathbf{C}^E_{C_j}$ for $i,j \in [\![1,p]\!]$ be any two elements of the set. Then the pair $\{\mathbf{C}^E_{C_i}, \mathbf{C}^E_{C_j}\}$ has both a greatest lower bound

$$\mathbf{C}^E_{C_i} \cap \mathbf{C}^E_{C_j} = \bigcap_{e\in E, S_{C_i}\subset e \vee S_{C_j}\subset e} e$$

and a least upper bound

$$\mathbf{C}^E_{C_i} \cup \mathbf{C}^E_{C_j} = \bigcap_{e\in E, S_{C_i}\subset e, S_{C_j}\subset e} e$$

Thus, any pair $\{\mathbf{C}^E_{C_i}, \mathbf{C}^E_{C_j}\}$ has a *glb* and a *lub*. Hence, the set of all $\mathbf{C}^E_{C_{i_1}\wedge\dots\wedge C_{i_k}}$ for $i_1 \dots i_k \in [\![1,p]\!]$ is a lattice. $\diamond$

Least Element.

We now prove that $\mathbf{C}^E_{C_i\wedge C_j}$ is the least element for $\mathbf{C}^E_{C_i}, \mathbf{C}^E_{C_j}$, that is $\mathbf{C}^E_{C_i\wedge C_j}$ is included in $\mathbf{C}^E_{C_i} \cap \mathbf{C}^E_{C_j}$. Let us first prove that $\mathbf{C}^E_{C_i\wedge C_j} \subseteq \mathbf{C}^E_{C_i}$. As $S_{C_i\wedge C_j} = \{(s_1 \dots s_n) \in D, C_i(s_1 \dots s_n) \wedge C_j(s_1 \dots s_n)\}$ and $S_{C_i} = \{(s_1 \dots s_n) \in D, C_i(s_1 \dots s_n)\}$, then $S_{C_i\wedge C_j} \subseteq S_{C_i}$. Thus $\bigcap_{e\subseteq E, S_{C_i\wedge C_j}\subseteq e} e \subseteq \bigcap_{e\subseteq E, S_{C_i}\subseteq e} e$. Hence $\mathbf{C}^E_{C_i\wedge C_j} \subseteq \mathbf{C}^E_{C_i}$. Similarly, we prove that $\mathbf{C}^E_{C_i\wedge C_j} \subseteq \mathbf{C}^E_{C_j}$. As $\mathbf{C}^E_{C_i\wedge C_j} \subseteq \mathbf{C}^E_{C_i}$ and $\mathbf{C}^E_{C_i\wedge C_j} \subseteq \mathbf{C}^E_{C_j}$, then $\mathbf{C}^E_{C_i\wedge C_j} \subseteq \mathbf{C}^E_{C_i} \cap \mathbf{C}^E_{C_j}$. Therefore $\mathbf{C}^E_{C_i\wedge C_j}$ is the least element for $\mathbf{C}^E_{C_i}, \mathbf{C}^E_{C_j}$. $\diamond$

$\square$

This proposition provided that each constraint of the CSP comes with a propagator, it is sufficient to apply these propagators iteratively, no matter the order, until the fixpoint is reached, to achieve consistency for the CSP. On existing consistencies, a similar idea has been proposed in [APT 99] or in [BEN 96].

Example 3.2.1 – Consider the CSP on real variables v_1, v_2 on domains $D_1 = [0,5]$, $D_2 = [0,5]$ and the constraints

$C1$: $5v_1 + v_2 \geq 10$
$C2$: $2v_1 + 5v_2 \leq 20$
$C3$: $2v_2 - v_1 \geq 1$

Figure 3.1 shows the lattice with inclusion of this CSP $\mathbb{B}$-consistencies. For each element in this lattice, we have in dark pink the set of solutions and in light pink the set approximated by the $\mathbb{B}$-consistency represented by a green box. The dotted black boxes are only there to framed the elements. For the elements corresponding to intersections, the dashed boxes correspond to the boxes in the intersection and the result is a plain box.

We can see that $\forall i_1, \ldots, i_k \in [\![1,3]\!], \mathbf{C}^{\mathbb{B}}_{C_{i_1} \wedge \cdots \wedge C_{i_k}} \subseteq \mathbf{C}^{\mathbb{B}}_{C_{i_1}} \cap \cdots \cap \mathbf{C}^{\mathbb{B}}_{C_{i_k}} \subseteq \mathbf{C}^{\mathbb{B}}_{C_{i_1}}, \ldots, \mathbf{C}^{\mathbb{B}}_{C_{i_k}}$. For instance, we have $\mathbf{C}^{\mathbb{B}}_{C_1 \wedge C_3} \subseteq \mathbf{C}^{\mathbb{B}}_{C_1} \cap \mathbf{C}^{\mathbb{B}}_{C_3} \subseteq \mathbf{C}^{\mathbb{B}}_{C_1}, \mathbf{C}^{\mathbb{B}}_{C_3}$.

3.2.2. *Splitting Operator*

Once the consistent element computed, we still need to explore it in order to find the solutions. This is done by cutting the remaining search space into smaller elements. An element is cut using a splitting operator as defined below.

Definition 3.2.3 (Splitting Operator in E)**.** Let $(E, \subseteq)$ be a partially ordered set. A *splitting operator* is an operator $\oplus : E \to \mathcal{P}(E)$ such that $\forall e \in E$,

1) $|\oplus(e)|$ is finite, $\oplus(e) = \{e_1, \ldots, e_k\}$,

2) $\bigcup_{i \in [\![1,k]\!]} e_i = e$,

3) $\forall i \in [\![1,k]\!], e \neq \emptyset \implies e_i \neq \emptyset$,

4) $\exists i \in [\![1,k]\!], e_i = e \implies e$ is a least element of E^f.

The first condition is needed for the search process to terminate. Indeed, the width of the search tree must be finite so that the search tree can be. The second condition ensures that the solving process is complete, in other words, that the splitting operator does not loose solutions. Moreover, this condition ensures that no element is added. The third condition forbids the splitting operator to return an empty domain. The fourth condition means that the splitting operator actually splits: it is forbidden to keep the same domains.

Remark 3.2.2 – It is important to notice that the definition implies that if e is not a least element of E^f, then $\forall i \in [\![1,k]\!], e_i \subsetneq e$.

Remark 3.2.3 – Also note that this definition of the splitting operator does not ask for $\oplus(e)$ to be a partition of e. This is mandatory in order to include the split on intervals with floating-point bounds. This does not affect the completeness of the solving process. As shown on Figure 3.2, a splitting operator is applied to the element on the left (3.2(a)), giving us the two elements on the right (3.2(b)). We can see that the starting element is included in the union of the elements obtained with the splitting operator and that the intersection of these elements is not empty.

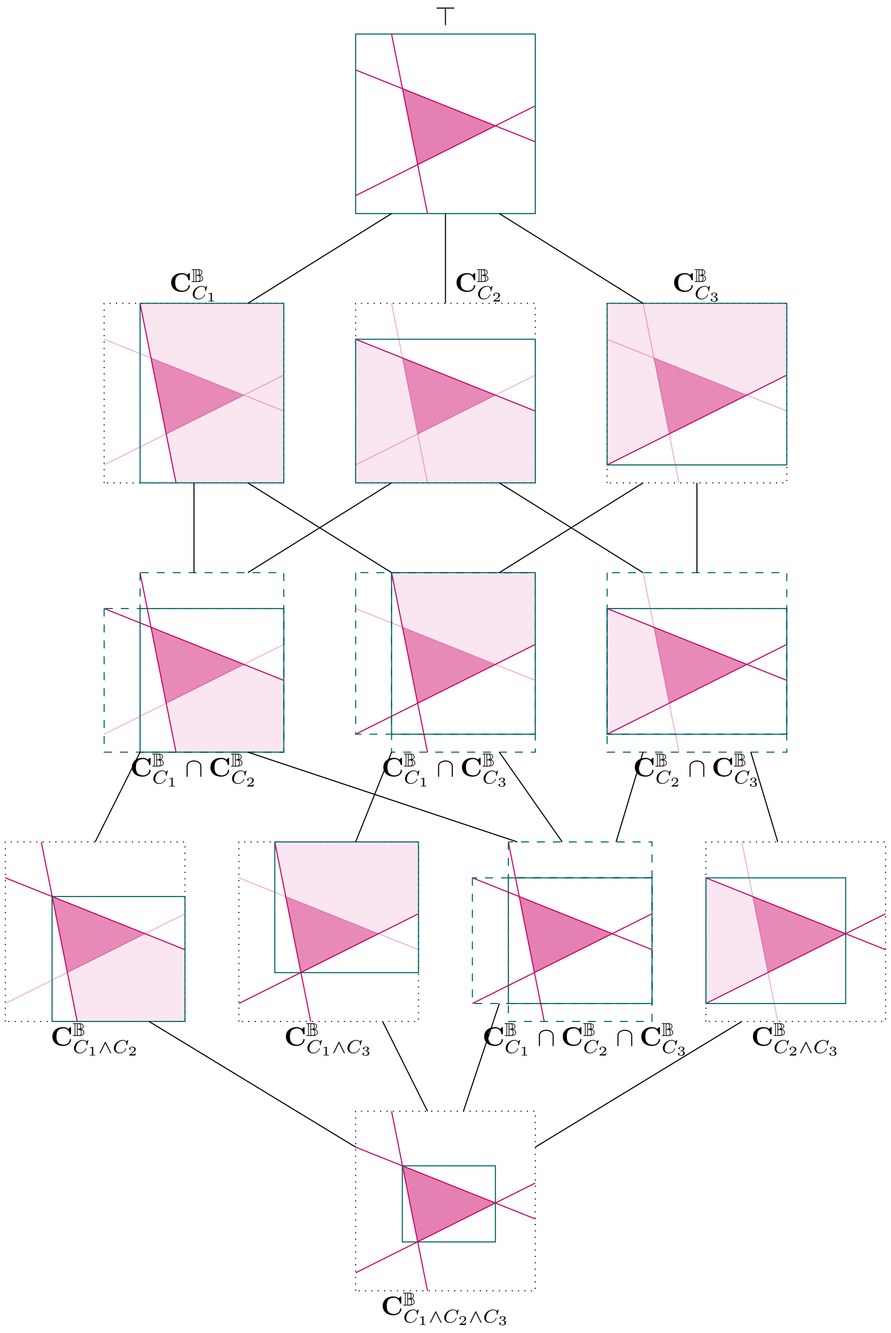

$\top$
$\mathbf{C}^{\mathbb{B}}_{C_1}$
$\mathbf{C}^{\mathbb{B}}_{C_2}$
$\mathbf{C}^{\mathbb{B}}_{C_3}$
$\mathbf{C}^{\mathbb{B}}_{C_1} \cap \mathbf{C}^{\mathbb{B}}_{C_2}$
$\mathbf{C}^{\mathbb{B}}_{C_1} \cap \mathbf{C}^{\mathbb{B}}_{C_3}$
$\mathbf{C}^{\mathbb{B}}_{C_2} \cap \mathbf{C}^{\mathbb{B}}_{C_3}$
$\mathbf{C}^{\mathbb{B}}_{C_1 \wedge C_2}$
$\mathbf{C}^{\mathbb{B}}_{C_1 \wedge C_3}$
$\mathbf{C}^{\mathbb{B}}_{C_1} \cap \mathbf{C}^{\mathbb{B}}_{C_2} \cap \mathbf{C}^{\mathbb{B}}_{C_3}$
$\mathbf{C}^{\mathbb{B}}_{C_2 \wedge C_3}$
$\mathbf{C}^{\mathbb{B}}_{C_1 \wedge C_2 \wedge C_3}$

Figure 3.1. *Example of finite lattices for the inclusion, which elements are the* $\mathbb{B}$*-consistencies of the CSP given in Example 3.2.1.*

We show below that the discrete instantiation and the continuous split are included in the definition. We will use the following notation for Cartesian domains: let $\oplus_1 : E_1 \to \mathcal{P}(E_1)$ be an operator for a partially ordered set E_1. Let E_2 be another partially ordered set and Id the identity function on E_2. We write $\oplus_1 \times Id$ the operator on $E_1 \times E_2$ such that $\oplus_1 \times Id(e_1, e_2) = \cup_{e \in \oplus_1(e_1)} e \times e_2$. We also write Id^i for the Cartesian product of i times Id.

Example 3.2.2 – Instantiation of a discrete variable is a splitting operator on $\mathcal{P}(\mathbb{N})$: $\oplus_{\mathbb{N}}(d) = \cup_{v \in d}\{v\}$. For every $i \in [\![1, n]\!]$, the operator $\oplus_{\mathbb{N}^n,i}(d) = Id^{i-1} \times \oplus_{\mathbb{N}} \times Id^{n-i-1}$, where $\oplus_{\mathbb{N}}$, at the i-th place, is a splitting operator. This consists in choosing a variable v_i and a value v in D_i.

Example 3.2.3 – The continuous splitting operator is defined on $\mathcal{P}(\mathbb{I}) : \oplus_{\mathbb{I}}(I) = \{I^{\vdash}, I^{\dashv}\}$ with $I^{\vdash}$ and $I^{\dashv}$ two non empty subintervals of I such that $I^{\vdash} \cup I^{\dashv} = I$ with whatever way of managing the rounding errors on floats. The most usual splitting operator for continuous CSP, for $I = [a, b]$, returns $I^{\vdash} = [a, h]$ and $I^{\dashv} = [h, b]$ with $h \in \mathbb{F}, h = \frac{a+b}{2}$ rounded in any direction. To ensure that the splitting operator terminates, it stops when a and b are two consecutive floats.

Example 3.2.4 – The usual splitting operator for a Cartesian product of intervals is defined in $\mathcal{P}(\mathbb{B})$. Let $I \in \mathbb{B}, D = I_1 \times \cdots \times I_n$. The splitting operator first chooses an interval $I_i, i \in [\![1, n]\!]$. Then this interval is split using $\oplus_{\mathbb{I}}$. We thus have:

$$\oplus_{\mathbb{B}}(I) = \{I_1 \times \cdots \times I_i^{\vdash} \times \cdots \times I_n, I_1 \times \cdots \times I_i^{\dashv} \times \cdots \times I_n\}$$

with $\{I_i^{\vdash}, I_i^{\dashv}\} = \oplus_{\mathbb{I}}(I_i)$. The most common continuous splitting operator in Constraint Programming, generally splits the domain maximizing $\max\limits_{i \in [\![1,n]\!]} (\overline{I_i} - \underline{I_i})$.

In the following, we will write, for any E, $\oplus_E$ the splitting operator in E. Thanks to these generic definitions of consistency and splitting operator, we can now define abstract domains in Constraint Programming.

3.2.3. *Abstract Domains*

Given that our goal is to define a generic solver totally independent from the domains representation, we define in this section the abstract domains for Constraint Programming. Those are defined such that they have the requirements to be part of a solving process.

Definition 3.2.4 (Abstract Domain for Constraint Programming)**.** An *abstract domain* is defined by:

– a complete lattice E,

– a concretization $\gamma : E \to D$, and an abstraction $\alpha : D \to E$ forming a Galois connection between E and the search space,

– a computer representable normal form,

– a sequence of splitting operators for E,

– a monotonic size function $\tau : E \to \mathbb{R}^+$ such that $\tau(e) = 0 \Leftrightarrow e = \emptyset$.

The Galois connection connects the abstract domain to a computer representation of the CSP domains. However, as for the abstract domains in Abstract Interpretation, it may not exist. The normal form ensures that there exists a unique representation for each abstract domain and that they can be comparable. The size function τ represents some measure of the abstract domain. It is an artificial technique to express the precision of an abstract domain. It is used later as a termination criterion in the solving method.

Abstract domains can be defined independently from the domains of a particular CSP. They are intended to represent the shape of the domains representation. Of course, they can be cartesian, but this is no longer mandatory. Note that we do not formalize the propagators as a part of an abstract domain. Because they depend both on the constraints and on the shape of the abstract domain, they have to be *ad hoc*.

With this definition, one can, for instance, define a Shadok[1] abstract domain. Given that they find a way to compute it, that they are computer representable and that there exist a Shadok consistency computing, if it exists, the smallest Shadok containing the solutions. Figure 3.2(b) shows the result of a possible splitting operator.

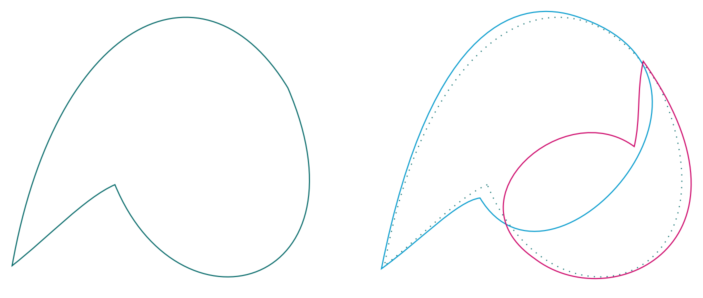
As the abstract domains are now defined for Constraint Programming, a solving method based on abstract domains can be defined.

3.3. Unified Solving

With the definitions of consistency (Definition 3.2.1) and splitting operator (Definition 3.2.3) that no longer depend on the domains representation, and with the definition of abstract domains in Constraint Programming (Definition 3.2.4), it is now possible to define an unified solving method based on abstract domains. This method is defined as the classical iteration of propagations and splits, but is no longer dedicated to a representation.

Algorithm 3.1 gives the pseudo-code of a solving method based on abstract domains. This algorithm looks a lot like the one for the continuous solving method

1. `http://www.lesshadoks.com/` or `http://en.wikipedia.org/wiki/Les\_Shadoks`



(a) a Shadok (b) an example of a Shadok split

Figure 3.2. *One can define a Shadok abstract domain. A Shadok in its normal form is given (3.2(a)) as well as the result of a possible splitting operator (3.2(b))*

```
list of abstract domains sols $\leftarrow \emptyset$                /* stores the abstract solutions */
queue of abstract domains toExplore $\leftarrow \emptyset$    /* stores the abstract elements to explore */
abstract domain $e \in E$                       /* data structure for the abstract domain */

$e = \alpha(\hat{D})$                                 /* initialization with initial domains */
push $e$ in toExplore

while toExplore $\neq \emptyset$ do
    $e \leftarrow$ pop(toExplore)
    $e \leftarrow E$-consistance($e$)
    if $e \neq \emptyset$ then
        if $\tau(e) \leq r$ or $e \subseteq S$ then
            sols $\leftarrow$ sols $\cup\, e$
        else
            choose a splitting operator $\oplus_E$
            push $\oplus_E(e)$ in toExplore
        end if
    end if
end while
```

Algorithm 3.1: Solving with abstract domains.

(Algorithm 2.2). Indeed, if one replaces everywhere in the pseudo-code "abstract domain" by "box", the continuous solver is obtained. The same apply for the integer cartesian product.

The following proposition gives the hypothesis under which this algorithm is complete and terminates.

Proposition 3.3.1. *If E is closed by intersection (H1), has no infinite decreasing chain (H2), and if $r \in \tau(E^f)$ (H3), then the solving process in Figure 3.1 terminates and is complete, in the sense that no solution is lost.*

Proof.

Termination.

We first prove that this algorithm terminates. Suppose that the search tree is infinite. By Definition 3.2.3 of the splitting operator, the width of the search tree is finite. Thus, it exists an infinite branch. Along this branch abstract domains are strictly decreasing as long as e is not the least element of E^f. By hypothesis (H2), there is a K such that $\forall k \geq K, e_k = e_K$. Let us study the different possible cases:

– if $e_K = \emptyset$ then the algorithm terminates as e_K contains no solution,

– if $e_K \neq \emptyset$ and $(\tau(e_K) \leq r$ or $e_K \subset S)$, then the algorithm terminates as e_K is a solution or contains no solution,

– if $e_K \neq \emptyset$ and $\tau(e_K) > r$ then e_K is split and $e_K \neq e_{K+1}$ which contradicts the definition of K.

We can thus conclude that the solving process in Algorithm 3.1 terminates. $\diamond$

Completeness.

As the splitting operator (Definition 3.2.3) and the consistency (Definition 3.2.1) are complete, the Algorithm 3.1 is complete. $\diamond$

Under hypothesis (H1), (H2) and (H3), the solving Algorithm 3.1 terminates and is complete.

$\square$

This proposition defines a generic solver on any abstract domain E with some hypothesis. Indeed, hypothesis (H1) and (H2) must be true, and r well chosen. If r is too large, the abstract element returned as solutions by the algorithm will be too large and can contain a large number of non solution points. On the contrary, if r is too small, the solving can take a long time and even worst, in the case of integer cartesian products, given the precision function τ, solutions may not be found. The efficiency of this solver will of course depend on the consistency algorithms on E and on the computer representation of the chosen abstract domain.

Usual solving algorithm in Constraint Programming are included in this definition as shown below.

Example 3.3.1 – For a fixed n, the set $\mathbb{S}$ with the splitting operator $\oplus_{\mathbb{N}^n,i}$ and the precision $\tau_{\mathbb{S}}(e) = \max(|X_i|)$ for $i \in [\![1, n]\!]$ is an abstract domain satisfying (H1) and (H2). In order to model the fact that the search process ends when all the variables are instantiated, one can take $r = 1$. As mentioned earlier, the $\mathbb{S}$-consistency is the generalized arc-consistency.

Example 3.3.2 – For a fixed n, the set $\mathbb{IB}$ with the splitting operator $\oplus_{\mathbb{N}^n,i}$ and the precision $\tau_{\mathbb{IB}}(e) = \max(b_i - a_i)$ for $i \in [\![1, n]\!]$ is an abstract domain satisfying (H1) and (H2). In order to model the fact that the search process ends when all the variables are instantiated, one can take $r = 1$. As mentioned earlier, the $\mathbb{IB}$-consistency is the bound-consistency.

Example 3.3.3 – For a fixed n, the set $\mathbb{B}$ with the splitting operator $\oplus_{\mathbb{B},i}$ and the precision $\tau_{\mathbb{B}}(I) = \max(\overline{I_i} - \underline{I_i})$ for $i \in [\![1, n]\!]$ is an abstract domain satisfying (H1) and (H2). In order to model the fact that the search process ends when a precision r is reached, one can stops when $\tau_{\mathbb{B}} \leq r$. Solving with $\mathbb{B}$ corresponds to the generally used solving process in continuous solvers with Hull-consistency.

These three examples show how to retrieve the usual abstract domains, which are cartesian and of a single type (integer cartesian products or intervals). It is now possible to define an abstract solver as an alternation of propagations and splits, see Figure 3.1. This solver returns an array of abstract domains containing only solution ($e \subseteq S$) or whose precision is less than r. The union of all of these elements is an approximation of the solution set.

3.4. Conclusion

In this chapter, we showed that it is possible to define a solver that is independent of the domains representation and therefore of the variables type of the problem. This solver includes usual Constraint Programming solving methods, regardless of the variables type (discrete or continuous). Furthermore, the domains are no longer limited to Cartesian representations. With this new definition, it is now possible to use different representations for domains. There exists a large number of abstract domains in Abstract Interpretation, such as the polyhedra, ellipsoids and zonotopes to name a fez. In the next chapter, we describe a non-Cartesian representation: the octagons.

Chapter 4

Octagons

In the generic solver presented in the previous chapter, abstract domains can be used to solve constraints satisfaction problems. In the same way as in Abstract Interpretation, we define the octagon abstract domain for Constraint Programming. This abstract domain already exists in Abstract Interpretation, it has been introduced by Miné [MIN 06]. We first define a representation that is computer representable, a splitting operator and a precision function. Then a Galois connection between the octagons and the boxes is given.

4.1. Definitions

In geometry, an octagon is, in $\mathbb{R}^2$, a polygon with eight sides [1]. As part of this work, we use a more general definition that was introduced in [MIN 06]. In the following, the term "octagon" is used to denote the octagons as defined below, when we want to talk about octagons as described in geometry, the term "mathematical octagon" will be used.

Definition 4.1.1 (Octagonal Constraint)**.** Let v_i, v_j be two variables. We call *octagonal constraint* constraints of the form $\pm v_i \pm v_j \leq c$ with $c \in \mathbb{R}$ a constant.

Remark 4.1.1 – Interval constraints ($v_i \geq a$, $v_i \leq b$) are particular cases of octagonal constraints.

Remark 4.1.2 – In a conjunction of constraints, two octagonal constraints are said redundant if and only if they have the same left side. For instance, the following

1. http://mathworld.wolfram.com/Octagon.html

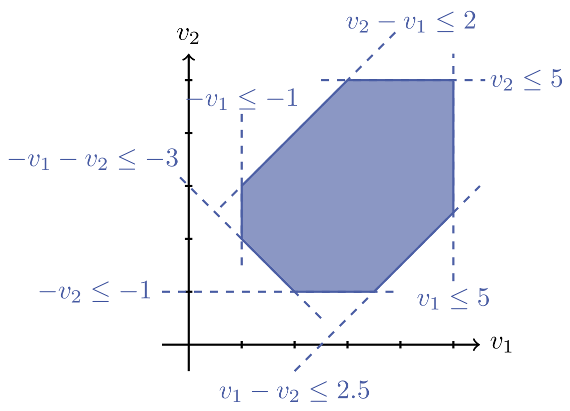


Figure 4.1. *Example of an octagon in* $\mathbb{R}^2$.

constraints $v_1 - v_2 \leq c$ and $v_1 - v_2 \leq c'$ are redundant. Only one of these constraints is effective, the one with the smallest constant (the smallest right side).

For instance in $\mathbb{R}^2$, octagonal constraints define straight lines which are parallel to the axis if $i = j$ and diagonal if $i \neq j$. This stays true in $\mathbb{R}^n$, where octagonal constraints define half-spaces (cut by hyperplanes). Consider a cube in $\mathbb{R}^3$, by adding octagonal constraints ($\pm v_i \pm v_j \leq c$), the edges are going to be cut, but not the corners.

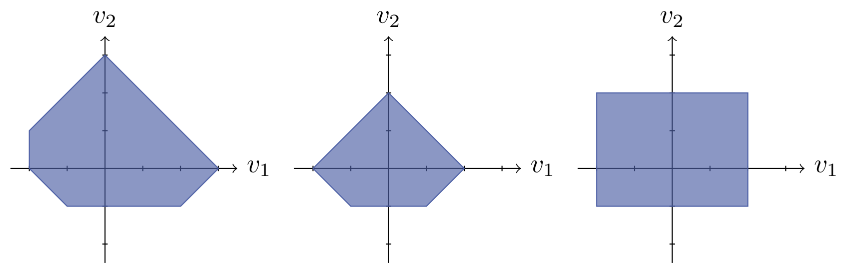
Definition 4.1.2 (Octagon)**.** An octagon is the set of points in $\mathbb{R}^n$ satisfying a conjunction of octagonal constraints.

Remark 4.1.3 – The geometric shape defined above includes the octagons but also other polygons. For instance, in $\mathbb{R}^2$ an octagon can have less than eight sides. In general, in $\mathbb{R}^n$, an octagon has at most $2n^2$ faces, which is the maximum number of possible non-redundant octagonal constraints on n variables. Moreover, octagons satisfying a conjunction of octagonal constraints are necessarily convex.

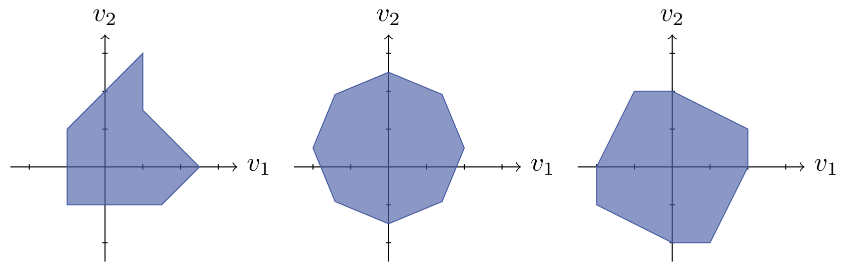
Also note that an octagon is a set of *real* points, but, like for the intervals, they can be restricted to have floating-points bounds ($c \in \mathbb{F}$). We thus have a real octagon with floating-point bounds.

An example of an octagon in $\mathbb{R}^n$ is given figure 4.1. We can see that an octagon can only have seven sides.

The first part of Figure 4.2 is composed of examples of octagons (4.2(a)). We can see that an octagon does not have necessarily 8 sides in 2 dimensions, and that



(a) Examples of octagons



(b) Examples of polygons that do not correspond to definition 4.1.2

Figure 4.2. *Examples of octagons 4.2(a) and polygons that do not respect the definition of octagons 4.2(b).*

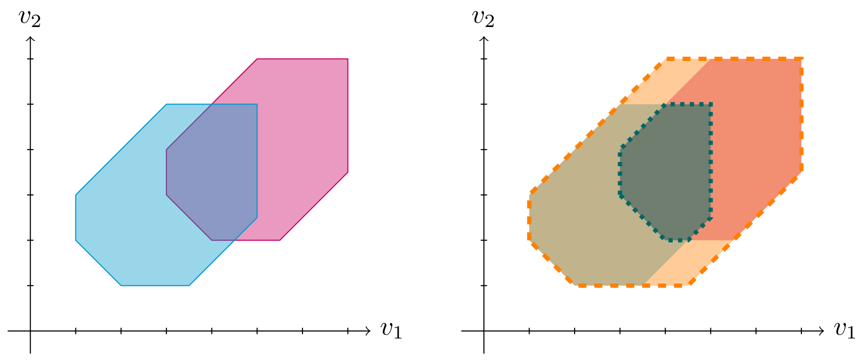
all the sides are parallel to the axis or to the diagonals. The second part (4.2(b)) shows polygons that do not correspond to the definition of octagons given previously (Definition 4.1.2). The first example has all its sides parallel to the axis or to the diagonals but is not convex. The next two exemples are mathematical octagons but their sides are not all parallel to the axis or the diagonals, they do not corresponds to Definition 4.1.2.

This definition of octagons offers interesting properties, that do not exist for mathematical octagons, such as the closure under intersection. Indeed, the intersection of any two octagons is still an octagon.

Remark 4.1.4 – The octagons are closed under intersection. Consider any two octagons $\mathbf{O} = \{\pm v_i \pm v_j \leq c\}$ and $\mathbf{O}' = \{\pm v_i \pm v_j \leq c'\}$ for $i, j \in [\![1, n]\!]$. Their intersection is also an octagon

$$\mathbf{O} \cap \mathbf{O}' = \{\pm v_i \pm v_j \leq \min(c, c')\}$$

Octagons are not closed under union, but the smallest octagon containing the union of any two octagons can easily be computed.



$$\left\{\begin{array}{rcr} -v_1 & \leq & -1 \\ v_1 & \leq & 5 \\ -v_2 & \leq & -1 \\ v_2 & \leq & 5 \\ -v_1 - v_2 & \leq & -3 \\ v_2 - v_1 & \leq & 2 \\ v_1 - v_2 & \leq & 2.5 \end{array}\right. \quad \left\{\begin{array}{rcr} -v_1 & \leq & -3 \\ v_1 & \leq & 7 \\ -v_2 & \leq & -2 \\ v_2 & \leq & 6 \\ -v_1 - v_2 & \leq & -6 \\ v_2 - v_1 & \leq & 1 \\ v_1 - v_2 & \leq & 3.5 \end{array}\right. \quad \overset{\text{union}}{\left\{\begin{array}{rcr} -v_1 & \leq & -1 \\ v_1 & \leq & 7 \\ -v_2 & \leq & -1 \\ v_2 & \leq & 6 \\ -v_1 - v_2 & \leq & -3 \\ v_2 - v_1 & \leq & 2 \\ v_1 - v_2 & \leq & 3.5 \end{array}\right.} \quad \overset{\text{intersection}}{\left\{\begin{array}{rcr} -v_1 & \leq & -3 \\ v_1 & \leq & 5 \\ -v_2 & \leq & -2 \\ v_2 & \leq & 5 \\ -v_1 - v_2 & \leq & -6 \\ v_2 - v_1 & \leq & 1 \\ v_1 - v_2 & \leq & 2.5 \end{array}\right.}$$

Figure 4.3. *Example of the union and the intersection of two octagons. The union gives the dashed orange octagon ant the intersection gives the dotted octagon.*

Remark 4.1.5 – Consider any two octagons $\mathbf{O} = \{\pm v_i \pm v_j \leq c\}$ and $\mathbf{O}' = \{\pm v_i \pm v_j \leq c'\}$ for $i, j \in [\![1, n]\!]$. The smallest octagon including the union is:

$$\mathbf{O} \cup \mathbf{O}' = \{\pm v_i \pm v_j \leq \max(c, c')\}$$

Figure 4.3 gives, geometrically and with conjunctions of constraints, the union and the intersection of the pink and the blue octagon. The orange octagon (dashed) corresponds to the smallest octagon including the union of the pink and blue octagons. The green octagon (dotted) corresponds to the intersection of the pink and blue octagons.

In the following, octagons are restricted to octagons with floating-point bounds ($c \in \mathbb{F}$). Without loss of generality, we assume octagons to be defined with no redundancies.

4.2. Representations

A necessary feature of abstract domains is that their elements must be computer representable. There exist no unique way to represent an abstract domain, however some representations are better suited for certain types of computations. We detail here two possible representations for octagons. The first one, the matrix representation, has been introduced in 1983 by Menasche and Berthomieu in [MEN 83]. It is used in [MIN 06]. The second one, the box representation, is part of the contribution of this work and is more suited for some computations in Constraint Programming, such as the consistency.

4.2.1. *Matrix Representation*

An octagon can be represented with a *difference bound matrix* (DBM) as described in [MEN 83, MIN 06]. This representation is based on a normalization of the octagonal constraints as follows.

Definition 4.2.1 (Difference Constraint)**.** Let w, w' be two variables. A *difference constraint* is a constraint of the form $w - w' \leq c$ with $c \in \mathbb{F}$ a constant.

By introducing new variables, it is possible to rewrite an octagonal constraint as an equivalent difference constraint. Let $C \equiv (\pm v_i \pm v_j \leq c)$ an octagonal constraint. New variables w_{2i-1}, w_{2i} are introduced such that w_{2i-1} corresponds to the positive form of v_i and w_{2i} to the negative form of v_i. In other words, $\forall i \in [\![1, n]\!], w_{2i-1} = v_i$ and $w_{2i} = -v_i$. Then:

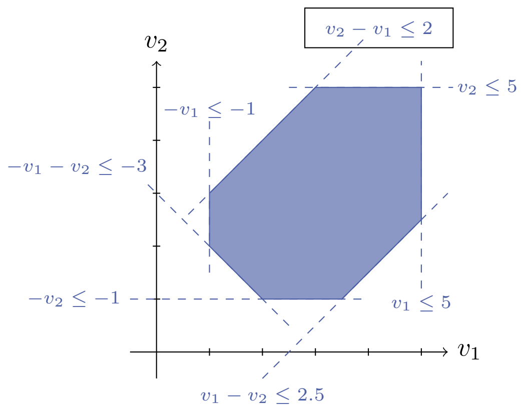
– for $i = j$

- if $C \equiv (v_i - v_i \leq c)$, then, if $c \geq 0, C$ is pointless, and can be removed; otherwise the corresponding octagon is empty and we stop. Indeed, there exists no value of v_i verifying the constraint $v_i - v_i < 0$,
- if $C \equiv (v_i + v_i \leq c)$, then C is equivalent to the difference constraint $(w_{2i-1} - w_{2i} \leq c)$,
- if $C \equiv (-v_i - v_i \leq c)$, then C is equivalent to the difference constraint $(w_{2i} - w_{2i-1} \leq c)$,

– for $i \neq j$

- if $C \equiv (v_i - v_j \leq c)$, then C is equivalent to the difference constraints $(w_{2i-1} - w_{2j-1} \leq c)$ and $(w_{2j} - w_{2i} \leq c)$,
- if $C \equiv (v_i + v_j \leq c)$, then C is equivalent to the difference constraints $(w_{2i-1} - w_{2j} \leq c)$ and $(w_{2j-1} - w_{2i} \leq c)$,
- if $C \equiv (-v_i - v_j \leq c)$, then C is equivalent to the difference constraints $(w_{2i} - w_{2j-1} \leq c)$ and $(w_{2j} - w_{2i-1} \leq c)$,
- if $C \equiv (-v_i + v_j \leq c)$, then C is equivalent to the difference constraints $(w_{2i} - w_{2j} \leq c)$ and $(w_{2j-1} - w_{2i-1} \leq c)$.

In the following, original variables are denoted by $(v_1 \ldots v_n)$ and the corresponding new variables by $(w_1, w_2, \ldots w_{2n})$ with $w_{2i-1} = v_i$ et $w_{2i} = -v_i$. Miné showed



(a)

$$\begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{array} \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 0 & -2 & \boxed{2} & -3 \\ 10 & 0 & +\infty & 2.5 \\ 2.5 & -3 & 0 & -2 \\ +\infty & \boxed{2} & 10 & 0 \end{pmatrix} \end{array}$$

(b)

Figure 4.4. *Equivalent representations of an octagon: with octagonal constraints 4.4(a) and with a difference bound matrix 4.4(b).*

in [MIN 06] that the difference constraints created by replacing positive and negative occurrences of the variable v_i by the corresponding w_i, represent the same octagon as the one obtained with octagonal constraints. Difference constraints can be stored in a difference bound matrix and octagonal constraints can be translate into difference constraints, hence, a difference bound matrix is a possible representation for octagons.

Definition 4.2.2 (Difference Bound Matrix)**.** Let **O** be an octagon in $\mathbb{F}^n$, and its set of difference constraints as defined previously. The *difference bound matrix* is a square matrix $2n \times 2n$ such that the element line i and column j is the constant c from the difference constraint $w_j - w_i \leq c$.

An example is given Figure 4.4(b), the octogone is given graphically (4.4(a)) and corresponds to the difference bound matrix (4.4(b)). The element on line 1 and column 3 and the element line 4 and column 2 correspond to the same constraint: $v_2 - v_1 \leq 2$.

At this stage, different difference bound matrices may represent the same octagon. For example Figure 4.4(b) the element line 2 column 3 can be replaced by 100 for instance without changing the corresponding octagon. In [MIN 06], an algorithm is defined to optimally compute the smallest values for the difference bound matrix. This algorithm is an adaptation of Floyd-Warshall shortest path algorithm [FLO 62]. It is modified in order to take advantages of the difference bound matrix structure. It exploits the fact that w_{2i-1} and w_{2i} correspond to the same variable.

Algorithm 4.1 gives the pseudo-code of the modified version of Floyd-Warshall shortest path algorithm for octagons. The instruction $i' \leftarrow (i \bmod 2 = 0)\,?\,i-1 : i+1$

```
float dbm[$2n$][$2n$]                    /* square difference bound matrix of $2n \times 2n$ */
int $i, j, k, i', j'$                          /* indexes for the difference bound matrix */

for $k$ from $1$ to $n$ do
  for $i$ from $1$ to $2n$ do
    for $j$ from $1$ to $2n$ do
      /* Computation of the shortest path */
      dbm[$i$][$j$] $\leftarrow$ min( dbm[$i$][$j$], dbm[$i$][$2k$] + dbm[$2k$][$j$],
                        dbm[$i$][$2k-1$] + dbm[$2k-1$][$j$],
                        dbm[$i$][$2k-1$] + dbm[$2k-1$][$2k$] + dbm[$2k$][$j$],
                        dbm[$i$][$2k$] + dbm[$2k$][$2k-1$] + dbm[$2k-1$][$j$] )
    end for
  end for

  /* Added to the original version */
  for $i$ from $1$ to $2n$ do
    for $j$ from $1$ to $2n$ do
      $i' \leftarrow (i$ mod $2 = 0)\,?\,i-1 : i+1$
      $j' \leftarrow (j$ mod $2 = 0)\,?\,j-1 : j+1$
      dbm[$i$][$j$] $\leftarrow$ min(dbm[$i$][$j$], dbm[$i$][$i'$] + dbm[$j'$][$j$])
    end for
  end for
end for
for $i$ from $1$ to $2n$ do
  if dmb[$i$][$i$] $< 0$ then
    return error
  else
    dbm[$i$][$i$] $\leftarrow 0$
  end if
end for
```

Algorithm 4.1: Modified version of Floyd-Warshall shortest path algorithm for the octagons.

signifies : if i is even then i' takes the value $i-1$, otherwise i' takes the value $i+1$. The beginning of this algorithm corresponds to Floyd-Warshall (computation of the shortest path part). The rest has been added and exploits the fact that w_{2i-1} and w_{2i} corresponds to the same variable v_i. Moreover, elements on the main diagonal must be greater or equal to zero, otherwise an error is raised because the corresponding octagon is empty. Indeed, $\forall i, w_i - w_i \leq c$, c cannot be less than 0. Another version of Algorithm 4.1 is proposed in [BAG 09]. These two versions have the same complexity ($O(n^3)$).

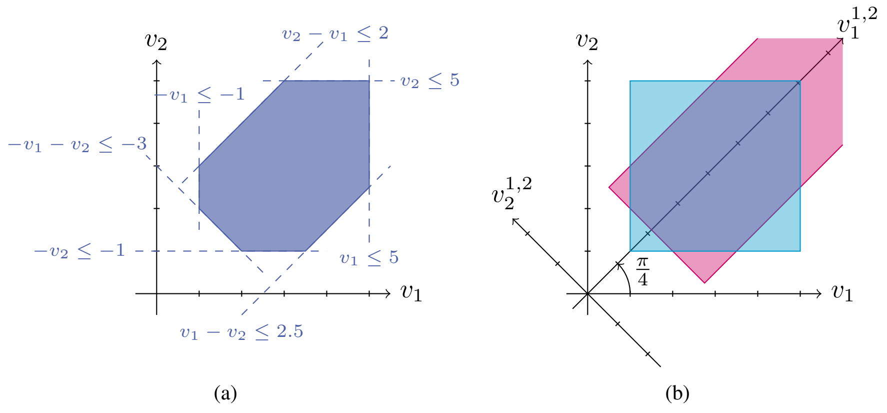


Figure 4.5. *Equivalent representations for the same octagon:the octagonal constraints 4.5(a) and the intersection of boxes 4.5(b).*

The execution of this modified version of Floyd-Warshall on the difference bound matrix given Figure 4.4(b) replaces both $+\infty$ by 10, which corresponds to adding the constraint $v_1 + v_2 \leq 10$ at the conjunction of octagonal constraints. Note that the implementation of the Floyd-Warshall algorithm replaces the $+\infty$ by 12.

We introduce a new representation for octagons based on the boxes (Definition 2.2.4). This representation combined with a difference bound matrix will be used to define, from an initial set of continuous constraints, an equivalent system taking into account the octagonal domains.

4.2.2. *Intersection of Boxes Representation*

In two dimensions, an octagon can be represented by the intersection of one box in the canonical basis for $\mathbb{F}^2$, and one box in the basis obtained from the canonical basis by a rotation of angle $\pi/4$. Figure 4.5 gives the representation by intersection of boxes (Figure 4.5(b)) corresponding to the octagon illustrated Figure 4.5(a). In this example, the octagon is define by 7 non redundant octagonal constraints, one of the boxes is thus unbound.

To generalize this notion to bigger dimensions, we introduce the notion of rotated basis as follows.

Definition 4.2.3 (Rotated basis)**.** Let $B = (u_1, \ldots, u_n)$ be the canonical basis of $\mathbb{F}^n$. Let $\alpha = \pi/4$. The *(i,j)-rotated basis* $B_\alpha^{i,j}$, for $i, j \in [\![1, n]\!]$, is the basis obtained after a rotation of α in the subplane defined by (u_i, u_j), the other vectors remaining unchanged:

$$B_\alpha^{i,j} = (u_1, \dots, u_{i-1}, (\cos(\alpha)u_i + \sin(\alpha)u_j), \dots u_{j-1}, (-\sin(\alpha)u_i + \cos(\alpha)u_j), \dots u_n)$$

By convention, for any $i \in [\![1, n]\!]$, $B_\alpha^{i,i}$ represents the canonical basis. In what follows, α is always $\pi/4$ and will be omitted. Finally, for $i, j, k \in [\![1, n]\!]$, every variable v_k living in the $B^{i,j}$ rotated basis and whose domain is D_k will be denoted by $v_k^{i,j}$ and its domain by $D_k^{i,j}$.

Remark 4.2.1 – Note that the idea of rotating variables and constraints has already been proposed in [GOL 10] in order to better approximate the solution set. However, their method is dedicated to under-constrained systems of equations.

The difference bound matrix can also be interpreted as a representation of the intersection of one box in the canonical basis and $n(n-1)/2$ other boxes, each one living in a rotated basis.

Let $\mathbf{O}$ be an octagon in $\mathbb{F}^n$ and M its difference bound matrix, with the same notations as above (M is a $2n \times 2n$ square matrix). For $i, j \in [\![1, n]\!]$, with $i \neq j$, let $\mathbf{B}_\mathbf{O}^{i,j}$ be the box $I_1 \times \dots \times I_i^{i,j} \times \dots \times I_j^{i,j} \dots \times I_n$, in the basis $B^{i,j}$, such that $\forall k \in [\![1, n]\!]$

$$\begin{aligned}
I_k &= [\ \underline{-\tfrac{1}{2}M[2k-1, 2k]}\ ,\ \overline{\tfrac{1}{2}M[2k, 2k-1]}] \\
I_i^{i,j} &= [\ \underline{-\tfrac{1}{\sqrt{2}}M[2j-1, 2i]}\ ,\ \overline{\tfrac{1}{\sqrt{2}}M[2j, 2i-1]}] \\
I_j^{i,j} &= [\underline{-\tfrac{1}{\sqrt{2}}M[2j-1, 2i-1]}\ ,\ \overline{\tfrac{1}{\sqrt{2}}M[2j, 2i]}]
\end{aligned}$$

Example 4.2.1 – Considering the difference bound matrix Figure 4.4(b), the boxes are $I_1 \times I_2 = [1, 5] \times [1, 5]$ et $I_1^{1,2} \times I_2^{1,2} = \left[\underline{3/\sqrt{2}}, +\infty\right] \times \left[\underline{-2.5/\sqrt{2}}, \overline{\sqrt{2}}\right]$, which correspond to the boxes Figure 4.5(b).

Proposition 4.2.1. *Let* $\mathbf{O}$ *be an octagon in* $\mathbb{F}^n$*, and* $\mathbf{B}_\mathbf{O}^{i,j}$ *the boxes as defined above. Then* $\mathbf{O} = \bigcap_{i,j \in [\![1,n]\!]} \mathbf{B}_\mathbf{O}^{i,j}$.

Proof.

Let $i, j \in [\![1, n]\!]$. We have $v_i^{i,j} = \frac{v_i+v_j}{\sqrt{2}}$ et $v_j^{i,j} = \frac{v_j-v_i}{\sqrt{2}}$ by Definition 4.2.3. Thus $(v_1 \dots v_i^{i,j} \dots v_j^{i,j} \dots v_n) \in \mathbf{B}_{\mathbf{O}}^{i,j}$ if and only if it satisfies the octagonal constraints on v_i and v_j, and the unary constraints for the other coordinates, in the difference bound matrix. The box $\mathbf{B}_{\mathbf{O}}^{i,j}$ is thus the solution set for these particular octagonal constraints. The points in $\bigcap_{i,j \in [\![1,n]\!]} \mathbf{B}_{\mathbf{O}}^{i,j}$ are exactly the points which satisfy all the octagonal constraints.

$\square$

To summarize, there exist different way to represent an octagon. An octagon can be represented using a difference bound matrix interpreted as a set of octagonal constraints (definition in intension). Or equivalently as an intersection of boxes (definition in extension). Moreover the conversion from one representation to the other is at the cost of a multiplication/division with the appropriate rounding mode.

We now have computer representable forms for the octagons. To define the octagon abstract domain, we need to define some operators, such as the splitting operator and the size function for the octagons. In the following, we suppose that the octagons are closed by the modified Floyd-Warshall algorithm, that they are only define by non-redundant octagonal constraints.

4.3. Abstract Domains Components

In the second phase of the solving process (exploration), if the current abstract element isn't considered as a solution, a splitting operator is used to cut the current abstract element into smaller abstract elements. To solve using octagons, a splitting operator is necessary, as well as a precision function to compute the size of an octagon. To define these operators, we use the intersection of boxes representation. Naturally these operators can be equivalently defined on the difference bound matrix.

4.3.1. *Octagonal Splitting Operator*

The octagonal splitting operator defined here extends the usual split operator to octagons. Splits can be performed in the canonical basis, thus being equivalent to the usual splits, or in a rotated basis. It can be defined as follows:

Definition 4.3.1 (Octagonal Splitting Operator)**.** Let $\mathbf{O}$ be an octagon defined as the intersection of boxes $I_1, \dots, I_n, I_1^{1,2}, \dots, I_1^{n-1,n}, \dots, I_n^{n-1,n}$, such that $\forall i, j, k \in [\![1, n]\!], I_k^{i,j} = [a, b]$ with $a, b \in \mathbb{F}$. The splitting operator $\oplus_o(\mathbf{O})$ for variable $v_k^{i,j}$ computes the two octagonal subdomains $I_1, \dots, [a, h], \dots, I_n^{n-1,n}$ and $I_1, \dots, [h, b], \dots, I_n^{n-1,n}$ with $h = \frac{a+b}{2}$ rounded in $\mathbb{F}$.

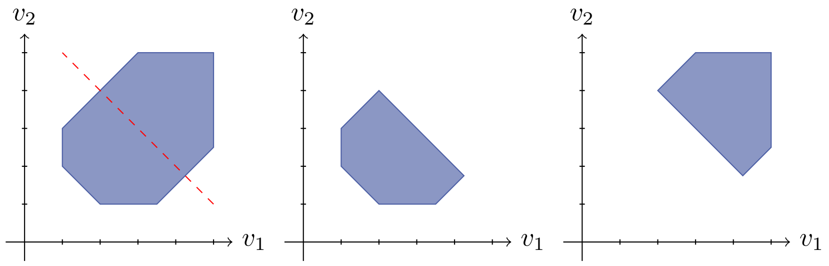


Figure 4.6. *Example of a split: the octagon on the left is cut in the $B^{1,2}$ basis.*

It is easily verified that the union of the two octagonal subdomains is equal to the original octagon. Thus this splitting operator does not lose solutions and corresponds to Definition 3.2.3. Note that this operator has three parameters i, j and k and not only one parameter i like in the interval splitting operator. However this definition does not take into account the correlation between the variables of the different basis. We take advantage again of the octagonal representation to communicate the domain reduction to the other basis. A split is thus immediately followed by a Floyd-Warshall propagation to update the octagonal constraints.

Figure 4.6 shows an example of the octagonal splitting operator. The octagon on the left is split allng the red dashed line.

4.3.2. *Octagonal Precision*

In most continuous solvers, the precision is defined as the size of the largest domain. For octagons, this definition leads to a loss of information because it takes the domains separately and does not take into account the correlation between the variables. In other words, this definition does not take into account that variables $v_1^{\{1,2\}}$ and $v_2^{\{1,2\}}$ depend on v_1 and v_2. Thus, we define an octagonal precision function taking the correlations of variables into account and corresponding to the diameter on all axis and diagonals.

Definition 4.3.2 (Octagonal Precision)**.** Let **O** be an octagon with its box representation $I_1 \dots I_n, I_1^{1,2} \dots I_n^{n-1,n}$. The octagonal precision is

$$\tau_o(\mathbf{O}) = \min_{i,j \in [\![1,n]\!]} \left(\max_{k \in [\![1,n]\!]} \left(\overline{I_k^{i,j}} - \underline{I_k^{i,j}} \right) \right)$$

For a single regular box, τ_o would be the same precision as usual. On an octagon, we take the minimum precision of the boxes in all the bases because it is more accurate. Moreover, this definition allows us to retrieve the operational semantics of the

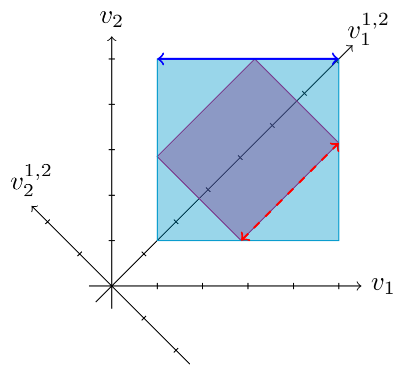


Figure 4.7. *Example of the octagonal precision: the usual continuous precision returns the the size of I_1 (the blue plain double arrow) while the octagonal precision returns the size of $I_1^{1,2}$ (the red dashed double arrow).*

precision, as shown by the following proposition: in an octagon of precision r over-approximating a solution set S, every point is at a distance at most r from S.

Proposition 4.3.1. *Consider a constraints satisfaction problem on variables ($v_1 \dots v_n$), with domains ($\hat{D}_1 \dots \hat{D}_n$), and constraints ($C_1 \dots C_p$). Let $\mathbf{O}$ be an octagon over-approximating the solution set S of this problem and containing at least a solution s. Let $r = \tau_o(\mathbf{O})$. Let $(x_1, \dots, x_n) \in \mathbb{F}^n$ be a point in $I_1 \times \cdots \times I_n, \forall i \in [\![1, n]\!], I_i \subseteq \hat{D}_i$. Then $\forall i \in [\![1, n]\!]$, $\min_{s \in S} |v_i - s_i| \leq r$, where $s = (s_1 \dots s_n)$. Each coordinate of all the points in $\mathbf{O}$ is at a distance at most r of a solution.*

Proof.

By Definition 4.3.2, the precision r is the minimum of some quantities in all the rotated basis. Let $B^{i,j}$ be the basis that realizes this minimum. The box $\mathbf{B}_{\mathbf{O}}^{i,j} = I_1 \times \cdots \times I_i^{i,j} \times \cdots \times I_j^{i,j} \cdots \times I_n$ is Hull-consistent and thus contains all the solutions S. Let $s \in S$. Because $r = max_k(\overline{I_k} - \underline{I_k}), \forall k \in [\![1, n]\!], |s_k - v_k| \leq \overline{I_k} - \underline{I_k} \leq r$. Hence, any point is at a distance at most r of a solution.

□

Figure 4.7 shows an example of the octagonal precision. Consider the octagon obtained with the intersections of the canonical box (in blue) and the box in the rotated basis (in pink). The usual precision function on boxes returns for this octagon the size of I_1. The octagonal precision function returns the size of $I_1^{1,2}$ which seems more adequate for the given octagon.

To define the octagon abstract domain in Constraint Programming, it only remains to prove that all octagons form a complete lattice and define a Galois connection between the octagons and floating-point bound boxes.

4.4. Abstract Domains

In this section, we first show that the set of octagons forms a complete lattice with inclusion.

Proposition 4.4.1. *The set of octagons forms a complete lattice $\mathbb{O}$.*

Proof.

Let $\mathbf{O}$ and $\mathbf{O}$' be any octagons. The set $\{\mathbf{O}, \mathbf{O}'\}$ has both a *lub* $\mathbf{O} \cup \mathbf{O}'$ (Remark 4.1.4) and a *glb* $\mathbf{O} \cap \mathbf{O}'$ (Remark 4.1.5). Thus, any finite set has both a *lub* and a *glb*. Hence $\mathbb{O}$ forms a complete lattice with inclusion.

□

We now show that there exists a Galois connection between octagons and floating-point bound boxes.

Proposition 4.4.2. *There exists a Galois connection* $\mathbb{O} \underset{\alpha_o}{\overset{\gamma_o}{\leftrightarrows}} \mathbb{B}$.

Proof.

– $\alpha_o : \mathbb{O} \to \mathbb{B}, \mathbf{O} \mapsto \{\pm v_i \leq c, i \in [\![1, n]\!]\}$ with $\mathbf{O} = \{\pm v_i \pm v_j \leq c\}$: only the constraints of the form $\pm v_i \pm v_i \leq c$ are kept, which correspond to the box bounds,

– $\gamma_o : \mathbb{B} \to \mathbb{O}, \mathbf{B} \mapsto \mathbf{B}$: is straightforward and exact as a box is an octagon.

The abstraction function α_o and the concretization function γ_o form a Galois connection.

□

From Proposition 4.4.1 and Proposition 4.4.2, we can say that octagons can be the base set of an abstract domain.

Definition 4.4.1 (Octagons Abstract Domain)**.** The octagon abstract domain is defined by:

– the complete lattice $\mathbb{O}$,

– the Galois connection $\mathbb{O} \underset{\alpha_o}{\overset{\gamma_o}{\leftrightarrows}} \mathbb{B}$,

– the representation with intersection of boxes,

– the octagonal splitting operator $\oplus_o$,

– the octagonal precision function τ_o.

Remark 4.4.1 – In this definition we have chosen the representation with intersection of boxes, but it can be replace with the matrix representation as they are equivalent.

This definition of octagons rely on the generation of all the possible basis $\frac{n(n+1)}{2}$. This transforms a CSP with n variables into a representation with n^2 variables. Among all these basis some are maybe less relevant than others. We present in the following a partial definition of the octagons.

Definition 4.4.2 (Partial Octagon)**.** Let $\mathcal{J}, \mathcal{K} \in \mathcal{P}(\llbracket 1, n \rrbracket)$ be sets of indexes. Let $\mathbf{O}$ be a set of octagonal constraints $\{\pm v_i \pm v_j \leq c \,|\, i \in \mathcal{J}, j \in \mathcal{K}\} \cup \{\pm v_i \pm v_i \leq c \,|\, i \in \llbracket 1, n \rrbracket\}$. The subset of $\mathbb{R}^n$ points satisfying all the constraints in $\mathbf{O}$ is called a partial octagon.

Example 4.4.1 – Let (v_1, v_2, v_3) be a set of variables. Let $\mathcal{J} = \{1\}$ and $\mathcal{K} = \{2, 3\}$. The partial octagon correspond to the set of points satisfying the following set of octagonal constraints $\{\pm v_1 \pm v_2 \leq c, \pm v_1 \pm v_3 \leq c'\} \cup \{\pm v_i \pm v_i \leq c \,|\, i \in \llbracket 1, n \rrbracket\}$.

The definition of partial octagons is very similar to the octagons. In both cases, it is a set of points satisfying a set of octagonal constraints. However, in the partial case, only a subset of all the possible non redundant octagonal constraints are expressed. Note that this definition includes the octagons. Indeed, when $\mathcal{J} = \mathcal{K} = \llbracket 1, n \rrbracket$, we have the Definition 4.1.2. Based on this observation, it can easily be shown that the properties of octagons hold for partial octagons, and hence that the set of partial octagons can be the base set for an abstract domain.

The choice of rotated basis to generate for a partial octagon depends on the problem to solve. Different octagonalization heuristics are presented Section 5.3.2.

4.5. Conclusion

In this chapter, we gave two equivalent and computer representable forms for the octagons. We have shown that the set of closed octagons by the modified version of Floyd-Warshall algorithm forms a complete lattice. Moreover, we have defined the octagonal splitting operator and size function as well as the Galois connection between the octagons and the boxes with floating-point bounds. With all these elements, we were able to define the octagons as an abstract domain in Constraint Programming. Since the octagon abstract domain relies on the generation of all the possible basis, we defined the partial octagons set. This set is the base set of an abstract domain. The next chapter gives additional details regarding the implementation of a solving method based on octagons.

Chapter 5

Octagonal Solving

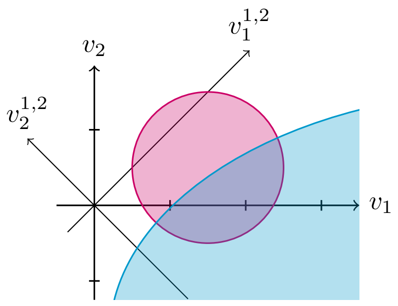
In this chapter we define a consistency based on octagons as well as a propagation schema. Thanks to this consistency and to the octagonal abstract domain defined in the previous chapter, we obtain a resolution method based on octagons. A prototype of this method has been implemented in a continuous solver. The details of this implementation are given in this chapter along with some preliminary results.

Consider a constraint satisfaction problem (CSP) on variables $(v_1 \dots v_n)$ in $\mathbb{R}^n$. The first step is to represent this problem in an octagonal form. We detail here the construction of an octagonal CSP from a CSP. We show that these two systems are equivalent.

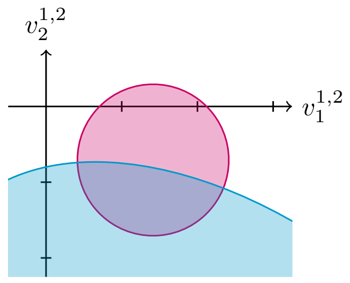
5.1. Octagonal CSP

First, the CSP is associated to an octagon, by stating all the possible non-redundant octagonal constraints $\pm v_i \pm v_j \leq c$ pour $i, j \in [\![1, n]\!]$. The constants c represent the bounds of the boxes in the rotated basis, or octagonal boxes, and the bound of the boxe in the canonical basis. These constants are dynamically modified during the solving process. They are initialized to $+\infty$.

The rotations defined in the previous chapter (Definition 4.2.3) introduce new axes, corresponding to new variables $v_i^{i,j}$ and $v_j^{i,j}$ in the (i, j)-rotated basis. These variables are redundant with the variables v_i and v_j in the canonical basis. If variables v_i and v_j are linked through constraints then the new variables $v_i^{i,j}$ and $v_j^{i,j}$ have to be linked as well. The CSP constraints $C_1 \dots C_p$ have to be rotated as well. We explain here how constraints are rotated.



(a) The constraints in the canonical basis.



(b) The constraints in $B^{1,2}$ the $(1,2)$-rotated basis.

Figure 5.1. *Example of rotated constraints: comparison between the initial CSP 5.1(a) and the rotated CSP 5.1(b).*

Given a function f on variables $(v_1 \dots v_n)$ in the canonical basis B, the expression of f in the (i,j)-rotated basis is obtained by symbolically replacing the i-th and j-th coordinates by their expressions in $B_\alpha^{i,j}$. Precisely, variable v_i is replaced by $\left(\cos(\alpha)v_i^{i,j} - \sin(\alpha)v_j^{i,j}\right)$ and variable v_j by $\left(\sin(\alpha)v_i^{i,j} + \cos(\alpha)v_j^{i,j}\right)$ where $v_i^{i,j}$ and $v_j^{i,j}$ are the coordinates for v_i and v_j in $B_\alpha^{i,j}$. The other variables are unchanged.

Definition 5.1.1 (Rotated constraint)**.** Given a constraint C holding on variables $(v_1 \dots v_n)$. The (i,j)-*rotated constraint* $C^{i,j}$ is the constraint obtained by replacing each occurrence of v_i by $\left(\cos(\alpha)v_i^{i,j} - \sin(\alpha)v_j^{i,j}\right)$ and each occurrence of v_j by $\left(\sin(\alpha)v_i^{i,j} + \cos(\alpha)v_j^{i,j}\right)$.

Example 5.1.1 – Let $v_1$and v_2 be two variables. Let C be the constraint $2v_1 + v_2 \leq 3$. The $(1,2)$-rotated constraint is:

$$C^{1,2} \equiv 2\left(\cos\left(\frac{\pi}{4}\right)v_1^{1,2} - \sin\left(\frac{\pi}{4}\right)v_2^{1,2}\right) + \left(\sin\left(\frac{\pi}{4}\right)v_1^{1,2} + \cos\left(\frac{\pi}{4}\right)v_2^{1,2}\right) \leq 3$$

given that $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, it can be simplified:

$$C^{1,2} \equiv 3v_1^{1,2} - v_2^{1,2} \leq 3\sqrt{2}$$

Figure 5.1 graphically compares the constraints in the canonical basis to the rotated ones. Figure 5.1(a) shows two constraints (pink and blue) like given in the initial CSP. The equivalent constraints after rotation of $\frac{\pi}{4}$ are given Figure 5.1(b). Rotated constraints live in the $(1,2)$-rotated basis $B^{1,2}$.

Given a continuous CSP on variables $(v_1 \dots v_n)$, with continuous domains $(\hat{D}_1 \dots \hat{D}_n)$ and constraints $(C_1 \dots C_p)$, we defined an octagonal CSP by adding the rotated variables, the rotated constraints and the rotated domains stored in a difference bound matrix.

To sum up, an octagonal CSP contains:

– the initial variables $(v_1 \dots v_n)$;

– the rotated variables $(v_1^{1,2}, v_2^{1,2}, v_1^{1,3}, v_3^{1,3} \dots v_n^{n-1,n})$, where $v_i^{i,j}$ is the i-th variable in the (i,j)-rotated basis $B_\alpha^{i,j}$;

– the initial constraints $(C_1 \dots C_p)$;

– the rotated constraints $(C_1^{1,2}, C_1^{1,3} \dots C_1^{n-1,n} \dots C_p^{n-1,n})$;

– the initial domains $(\hat{D}_1 \dots \hat{D}_n)$;

– a difference bound matrix which represents the rotated domains. It is initialized with the bounds of the regular domains $\hat{D}_i$ for the cells at position $(2i, 2i-1)$ and $(2i-1, 2i)$ for $i \in [\![1, n]\!]$, and $+\infty$ everywhere else.

In these conditions, the initial CSP is equivalent to this octagonal CSP, restricted to the variables $v_1 \dots v_n$, as shown in the following proposition.

Proposition 5.1.1. *Consider a CSP on variables $(v_1 \dots v_n)$, with domains $(\hat{D}_1 \dots \hat{D}_n)$ and constraints $(C_1 \dots C_p)$, and the corresponding octagonal CSP as defined above. The solution set of the original CSP S is equal to the solution set of the octagonal CSP restricted to variables $(v_1 \dots v_n)$.*

Proof.

Octagonal Solution $\Longrightarrow$ Solution.

First we prove that an octagonal solution is a solution in the original CSP. Let $s \in \mathbb{R}^n$ a solution of the octagonal CSP restricted to variables $(v_1 \dots v_n)$. We have $C_1(s) \dots C_p(s)$ and $s \in \hat{D}_1 \times \dots \times \hat{D}_n$. Hence s is a solution for the original CSP. $\diamond$

Solution $\Longrightarrow$ Octagonal Solution.

We now prove that a solution for the original CSP is an octagonal solution. Reciprocally, let $s \in \mathbb{R}^n$ a solution of the original CSP. The initial constraints $(C_1 \dots C_p)$ are true for s. Let us show that there exist values for the rotated variables such that the rotated constraints are true for s. Let $i, j \in [\![1, n]\!]$, $i \neq j$ and $k \in [\![1, p]\!]$. Let C_k be a constraint in the original CSP and $C_k^{i,j}$ the corresponding rotated constraint. By Definition 5.1.1 $C_k^{i,j}(v_1 \dots v_{i-1}, \cos(\alpha)v_i^{i,j} - \sin(\alpha)v_j^{i,j}, v_{i+1} \dots \sin(\alpha)v_i^{i,j} + \cos(\alpha)v_j^{i,j} \dots v_n) \equiv C_k(v_1 \dots v_n)$. Let us define the two reals $s_i^{i,j} = \cos(\alpha)s_i + \sin(\alpha)s_j$ and $s_j^{i,j} = -\sin(\alpha)s_i + \cos(\alpha)s_j$

the image of s_i and s_j by the rotation of angle α. By reversing the rotation, $\cos(\alpha)s_i^{i,j} - \sin(\alpha)s_j^{i,j} = s_i$ and $\sin(\alpha)s_i^{i,j} + \cos(\alpha)s_j^{i,j} = s_j$, thus $C_k^{i,j}(s_1 \ldots s_i^{i,j}, \ldots s_j^{i,j} \ldots s_n) = C_k(s_1 \ldots s_n)$ is true. It remains to check that $(s_1 \ldots s_i^{i,j}, \ldots s_j^{i,j} \ldots s_n)$ is in the rotated domain, which is true because the difference bound matrix is initialized at $+\infty$. Hence s is a solution for the octagonal CSP. $\diamond$

$\square$

For a CSP on n variables, this representation has an order of magnitude n^2. Indeed the octagonal CSP has n^2 variables and domains, and at most $p\left(\frac{n(n-1)}{2} + 1\right)$ constraints. Of course, many of these objects are redundant. We explain in the next sections how to use this redundancy to speed up the solving process.

As said when defining abstract domains in Constraint Programming Section 3.2.3, propagators have to be *ad hoc* as they depend both on the abstract domain and constraints. The next sections show how the consistency and propagation scheme can be defined for the octagons.

5.2. Octagonal Consistency and Propagation

First, we generalize the definition of Hull-consistency on intervals (Definition 2.2.9) to the octagons and define the propagators for the rotated constraints. Then we define an efficient propagation scheme for both the rotated constraints and octagonal constraints using the modified version of Floyd-Warshall algorithm (Algorithm 4.1).

5.2.1. *Octogonal Consistency*

We generalize the definition of Hull-consistency on intervals for any continuous constraint to the octagons. With the intersection of boxes representation, we show that any propagator for Hull-consistency on boxes can be extended to a propagator on the octagons. For a given n-ary relation on $\mathbb{R}^n$, there is a unique smallest octagon (w.r.t. inclusion) which contains the solutions of this relation, as shown in the following proposition.

Proposition 5.2.1. *Consider a constraint C (resp. a constraints sequence $(C_1 \ldots C_p)$), and S_C its set of solutions (resp. S). Then there exist a unique octagon $\mathbf{O}$ such that: $S_C \subseteq \mathbf{O}$ (resp. $S \subseteq \mathbf{O}$), and for all octagons $\mathbf{O}'$, $S_C \subseteq \mathbf{O}'$ implies $\mathbf{O} \subseteq \mathbf{O}'$. $\mathbf{O}$ is the unique smallest octagon containing the solutions, w.r.t. inclusion. $\mathbf{O}$ is said Oct-consistent for the constraint C (resp. constraints sequence $(C_1 \ldots C_p)$).*

Proof.

Octagons are closed by intersection (Remark 4.1.4) and the set of floating bound octagons forms a finite lattice and thus complete (Proposition 4.4.1). Hence the proof comes directly form Proposition 3.2.5.

□

Proposition 5.2.2. *Let C be a constraint, and $C^{i,j}$ the (i,j)-rotated constraint for $i,j \in [\![1,n]\!]$. Let $\mathbf{B}^{i,j}$ be the Hull-consistent box for $C^{i,j}$, and $\mathbf{B}$ the Hull-consistent box for C. The Oct-consistent octagon for C is the intersection of all the $\mathbf{B}^{i,j}$ and $\mathbf{B}$.*

Proof.

The intersection of boxes representation is an exact representation for octagons (Proposition 4.2.1). The proof comes directly form Proposition 5.2.1.

□

The consistency indicates how to filter the inconsistent values for a constraints, We now have to chose in which order the constraints are propagated.

5.2.2. *Propagation Scheme*

The modified version of Floyd-Warshall shortest path (Algorithm 4.1) efficiently and optimally propagates octagonal constraints [DEC 89]. The initial constraints $(C_1 \dots C_p)$ and the rotated constraints $(C_1^{1,2} \dots C_p^{n-1,n})$ are filtered using the Oct-consistency. So we need to integrate in the same propagation scheme both propagators. Octagonal constraints corresponding to the bounds of the boxes, we rely in the informations gathered during the propagation of these constraints to propagate the initial and rotated constraints. This allows us to take full advantage of relational properties of octagons. It is important to note that all the propagators defined above are monotonous and complete, we can then combine them in any order to compute the consistency, as shown in Proposition 3.2.5 or in [BEN 96].

The key idea for the propagation scheme is to interleave the refined Floyd-Warshall algorithm and the propagators for the initial and rotated constraints. A pseudocode is given Algorithm 5.1. During the first iteration of the propagation loop, all the propagators for the initial and rotated constraints are executed. This first stage reduces the domains with respect to the constraints of the problem. As the domains are stored in the difference bound matrix, the values in the matrix are thus modified and the modified version of Floyd-Warshall must be applied to compute the new minimal bounds. The second part of the propagation loop corresponds to Algorithm 4.1 except that every time that an element is modified, the corresponding propagators are added to

```
float dbm[$2n$][$2n$]        /* difference bound matrix containing the variables domain */
set of propagators propagSet        /* set of propagators to execute */
int $i, j, k, i', j'$        /* indexes for the difference bound matrix */
float $m$        /* auxiliary variable */

propagSet $\leftarrow \{\rho_{C_1}, \ldots, \rho_{C_p}, \rho_{C_1^{1,2}}, \ldots, \rho_{C_p^{n-1,n}}\}$
while propagSet $\neq \emptyset$ do
                        /* initial and rotated constraints propagation */
  apply all the propagators in propagSet
  propagSet $\leftarrow \emptyset$
                        /* octagonal constraints propagation */
  for $k$ from $1$ to $n$ do
    for $i$ from $1$ to $2n$ do
      for $j$ from $1$ to $2n$ do
        $m \leftarrow \min($ dbm[$i$][$2k$] $+$ dbm[$2k$][$j$], dbm[$i$][$2k-1$] $+$ dbm[$2k-1$][$j$],
                 dbm[$i$][$2k-1$] $+$ dbm[$2k-1$][$2k$] $+$ dbm[$2k$][$j$],
                 dbm[$i$][$2k$] $+$ dbm[$2k$][$2k-1$] $+$ dbm[$2k-1$][$j$]        $)$
        if $m <$ dbm[$i$][$j$] then
          dbm[$i$][$j$] $\leftarrow m$        /* update of the DBM */
          propagSet $\leftarrow$ propagSet $\cup \{\rho_{C_1^{i,j}} \ldots \rho_{C_p^{i,j}}\}$        /* get the propagators to
          apply */
        end if
      end for
    end for
    for $i$ from $1$ to $2n$ do
      for $j$ from $1$ to $2n$ do
        $i' \leftarrow (i$ mod $2 = 0)\ ?\ i-1 : i+1$
        $j' \leftarrow (j$ mod $2 = 0)\ ?\ j-1 : j+1$
        if (dbm[$i$][$i'$] $+$ dbm[$j'$][$j$])) $<$ dbm[$i$][$j$] then
          dbm[$i$][$j$] $\leftarrow$ dbm[$i$][$i'$] $+$ dbm[$j'$][$j$]
          propagSet $\leftarrow$ propagSet $\cup \{\rho_{C_1^{i,j}} \ldots \rho_{C_p^{i,j}}\}$
        end if
      end for
    end for
  end for
  for $i$ from $1$ to $2n$ do
    if dbm[$i$][$i$] $< 0$ then
      return failure
    else
      dbm[$i$][$i$] $\leftarrow 0$
    end if
  end for
end while
```

Algorithm 5.1: Pseudo code for the propagation loop mixing the modified version of Floyd Warshall algorithm (octagonal constraints propagation) and the initial and rotated propagators $\rho_{C_1} \ldots \rho_{C_p}, \rho_{C_1^{1,2}} \ldots \rho_{C_p^{n-1,n}}$, for an octagonal CSP as defined Section 5.1.

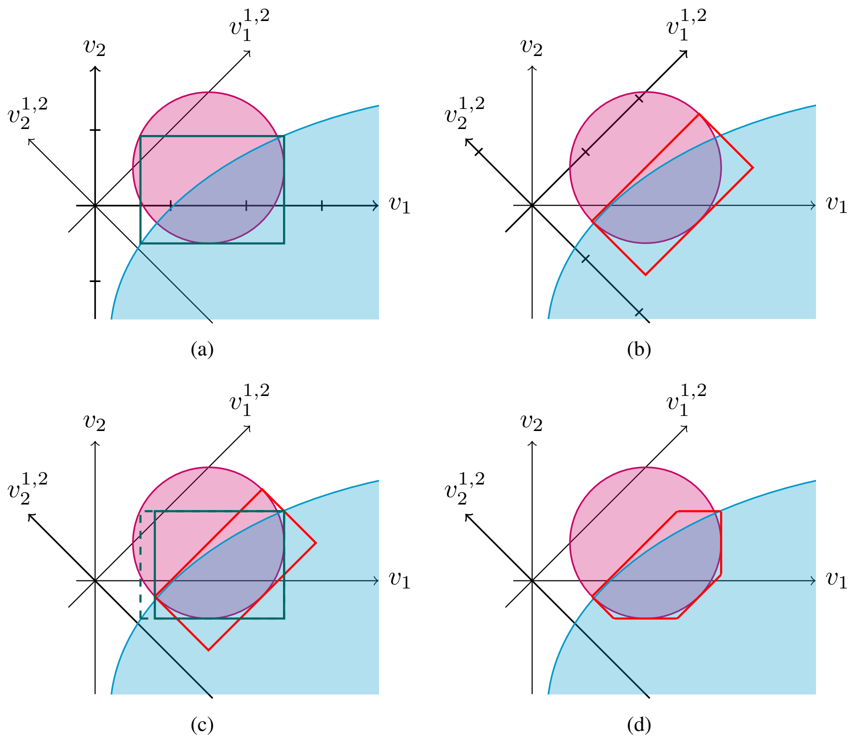


Figure 5.2. *Example of an iteration of the propagation for the Oct-consistency: an usual consistency algorithm is applied in each basis (Figures 5.2(a) and 5.2(b)) then the different boxes are made consistent using the modified Floyd-Warshall algorithm (Figure 5.2(c)). The consistent octagon is given Figure 5.2(d).*

the set of propagators to execute. The propagators in this set are applied once the propagation of the octagonal constraints is done. Hence, if an element is modified several times, its corresponding propagators are called only once. We can say that this propagation scheme is guided by the additional information of the relational domain.

An example of an iteration of the propagation loop is given Figure 5.2: propagators of the initial constraints ρ_{C_1}, ρ_{C_2} are first applied (Figure 5.2(a)), then the propagators of the rotated constraints are applied $\rho_{C_1^{1,2}}, \rho_{C_2^{1,2}}$ (Figure 5.2(b)). The two boxes are made consistent with respect to each other using the refined Floyd-Warshall algorithm(Figure 5.2(c)). In this example, the box in the canonical basis is slightly reduced, thus the propagators ρ_{C_1}, ρ_{C_2} have to be applied again. The application of these propagators do not modify the box in the canonical basis and the consistent octagon corresponds to the intersection of theses two boxes (Figure 5.2(d)).

We show here that the propagation as defined Algorithm 5.1 is correct, in other words, it computes the consistent octagon for a sequence of constraints.

Proposition 5.2.3 (Correctness)**.** *Given a CSP on variables ($v_1 \dots v_n$), domains ($\hat{D}_1 \dots \hat{D}_n$), and constraints ($C_1 \dots C_p$). Suppose that each constraint C comes with a propagator ρ_C such that ρ_C reaches the Hull-consistency, that is $\rho_C(\hat{D}_1 \times \dots \times \hat{D}_n)$ returns a hull-consistent box for C. Then the propagation scheme as define Algorithm 5.1 computes the Oct-consistent octagon for $C_1 \dots C_p$.*

Proof.

This proposition derives from Proposition 5.2.2, and the propagation scheme Algorithm 5.1. The propagation scheme is defined so as to stop when propagSet is empty. This happens when the difference bound matrix is no more modified. Indeed, the propagators are added to propagSet only when an element in the difference bound matrix is modified. Thus when the propagation scheme terminates, the octagonal constraints are consistent. Moreover, as the elements in the difference bound matrix are not changed, it follows that the propagators did not modified the variables domain. Hence the final octagon is stable by application of all $\rho_{C_k^{i,j}}$, for all $k \in [\![1, p]\!]$ and $i, j \in [\![1, n]\!]$. By hypothesis, the propagators reach consistency, the boxes are thus hull-consistent for all the initial and rotated constraints. By Proposition 5.2.2, the returned octagon is Oct-consistent.

□

The refined Floyd-Warshall has a time complexity of $O(n^3)$. For each round in its loop, in the worst case we add all the propagators to the set of propagators to execute, that is in the worst case $p\left(\frac{n(n-1)}{2} + 1\right)$ propagators. Thus the time complexity for the propagation scheme in Algorithm 5.1 is $O(n^3 + pn^2)$. In the end, the octagonal propagation uses both representations of octagons. It takes advantage both of the relational property of the octagonal constraints (Floyd-Warshall), and of the usual constraint propagation on boxes (propagators). This comes to the cost of computing the octagon, but is expected to give a better precision in the end.

Now that we have defined for the octagons all the necessary components for the solving process, namely the octagonal consistency, the octagonal splitting operator, the octagonal precision and the octagons abstract domain, we can go a step further and define a fully octagonal solver.

5.3. Octagonal Solver

We saw Section 4.4 that octagons can be the base set for an abstract domain and that with the consistency and the propagation loop described previously, we obtain an octagonal solver. The solving process is the same that the one in Algorithm 3.1 using octagons. Octagons being closed under intersection (H1) and $\mathbb{O}$ not having infinite decreasing chain (H2), this algorithm terminates and is complete. Moreover, from Algorithm 3.1, this algorithms returns a set of octagons which union over-approximates the solutions space. More precisely, an octagon is considered as a solution if all its points are solutions or if it over-approximates a solution set with a precision r.

To guide the search space exploration, one can define heuristics for the choice of the variable. Several heuristics are presented in the next subsection.

5.3.1. *Variables Heuristics*

An important feature of a solver is the heuristic used to choose the variable. It chooses which variable to split. For continuous constraints, the variable with the largest domain size is chosen to be split in two. by splitting the largest domain, this heuristic can arrive faster to the precision. The three following heuristics derived from this strategy and differ on the set of variables that can be chosen. Let V' the set of all the variables in an octagonal CSP $V' = (v_1 \dots v_n, v_1^{1,2}, v_2^{1,2} \dots\ v_{n-1}^{n-1,n}, v_n^{n-1,n})$ which is re-numbered $V' = (v_1\ \dots v_n, v_{n+1} \dots v_{n^2})$ for more simplicity.

Definition 5.3.1 (LargestFirst Heuristic (LF))**.** Let $V' = (v_1\ \dots v_n, v_{n+1} \dots v_{n^2})$ be the set of all the variables in the octagonal CSP. The variable to split is the variable v_i which realizes the maximum of

$$\operatorname*{arg\,max}_{i \in [\![1,n^2]\!]} \left(\overline{D_i} - \underline{D_i}\right)$$

As for the usual continuous split, the variable to split is chosen among all the variables.

Definition 5.3.2 (LargestCanFirst Heuristic (LCF))**.** Let $V' = (v_1\ \dots v_n, v_{n+1} \dots v_{n^2})$ be the set of all the variables in the octagonal CSP. The variable to split is the variable v_i realizing the maximum of

$$\operatorname*{arg\,max}_{i \in [\![1,n]\!]} \left(\overline{D_i} - \underline{D_i}\right)$$

In this strategy, the choice is restricted to the initial variables, that are the variables in the original CSP.

Definition 5.3.3 (LargestOctFirst Heuristic (LOF))**.** Let $V' = (v_1\ \dots v_n, v_{n+1} \dots v_{n^2})$ be the set of all the variables in the octagonal CSP. The variable to split is the variable v_i which realizes the maximum of

$$\operatorname*{arg\,max}_{i \in [\![n+1,n^2]\!]} \left(\overline{D_i} - \underline{D_i}\right)$$

In this strategy, the choice is restricted to the rotated variables, that are the variables generated by the rotations.

The three choice heuristics presented above do not take into account the correlation between the variables. Moreover, the variable which has the largest domain can be in a basis that is of little interest for the problem, or the domains have a wide range because the constraints are poorly propagated in this basis. We thus define an octagonal strategy.

Definition 5.3.4 (Oct-Split Heuristic (OS))**.** This strategy relies on the same remark as for definition 4.3.2: the variable to split is the variable $v_k^{i,j}$ which realizes the maximum of

$$\min_{i,j\in[\![1,n]\!]}\left(\max_{k\in[\![1,n]\!]}\left(\overline{D_k^{i,j}}-\underline{D_k^{i,j}}\right)\right)$$

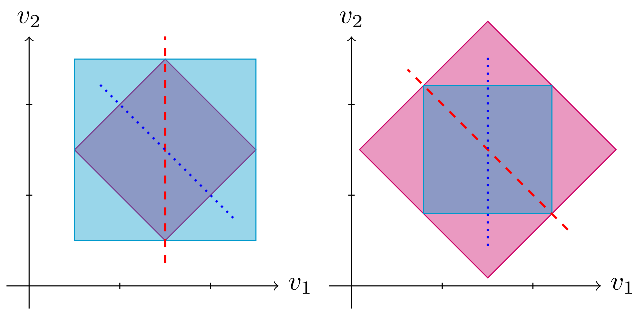
The strategy is the following: choose first a promising basis, that is, a basis in which the boxes are tight (choose i, j). Then take the worst variable in this basis as usual (choose k). This heuristic aims at reaching faster the octagonal precision (Definition 4.3.2).

Figure 5.3 shows examples of the different variables heuristics. In the first two examples, we consider the octagon obtain by the intersection of the blue and pink boxes. In the first example, strategies LargestFirst and LargestCanFirst chose to split in two the domain D_1 (along the red dashed line). While strategies LargestOctFirst and Oct-Split choose to split in two the domain $D_1^{1,2}$ (along the blue dotted line). In the second example, strategies LargestFirst and LargestOctFirst choose to split in two the domain $D_1^{1,2}$ (along the red dashed line). While strategies LargestCanFirst and Oct-Split choose to split in two the domain D_1 (along the blue dotted line).

For these two examples, the split along the blue dotted line seems the more relevant. In the first example, splitting along the blue dotted line reduces the size of the domains in the rotated basis which is interesting for this octagon as it corresponds to the box in the rotated basis. Similarly, in the second example, reducing the size of the domains in the canonical basis is more relevant as the octagon corresponds to the box in the canonical basis.

5.3.2. *Octogonalization Heuristic*

Octagonalization heuristics create partial octagons. For the octagons, all the $n(n+1)/2$ possible basis are generated, which for problems of high dimension (large value for n) created an octagon with a large number of faces. As said previously, maybe some of the generated basis are less interesting than others with respect to the problem to solve. We thus define different strategies to determine a subset of bases to generate.



(a) (b)

Figure 5.3. *Examples of the different variable heuristics. Figure 5.3(a), the LF and LCF strategies split along the red dashed line while the LOF and OS strategies split along the blue dotted line. Figure 5.3(b), the LF and LOF strategies split along the red dashed line while the LCF and OS strategies split along the blue dotted line.*

Given a problem we try to find, by symbolically looking at the constraints, which bases are more likely to improve the search. In other words, we try to determine in which bases the problem is easier to solve by only looking at the constraints. This step is done once at the creation of the octagon unlike in [GOL 10] where the basis changes after each call to the splitting operator in the solving process.

Definition 5.3.5 (ConstraintBased Heuristic (CB))**.** Consider a CSP on variables $(v_1 \dots v_n)$ to octogonalize. The basis $B_\alpha^{i,j}$ is generated only if the variables v_i and v_j appear within the same constraint.

If v_i and v_j appear in the constraint C, then C links v_i with v_j. We say that there exists a relation between v_i and v_j. The idea behind this heuristic is to emphasize a relation that already exists between two variables. In the worst case, all the pairs of variables appear within a constraint thus all the basis will be generated.

Definition 5.3.6 (Random Heuristic (R))**.** Consider a CSP on variables $(v_1 \dots v_n)$ to octogonalize. Among all the possible bases generates *one* random basis. Chooses i and j with $i < j$ then generates the basis $B_\alpha^{i,j}$.

We set the number of generated bases to one, this is purely arbitrary and can be random for a pure random heuristic.

Definition 5.3.7 (StrongestLink Heuristic (SL))**.** Consider a CSP on variables $(v_1 \dots v_n)$ to octogonalize. This heuristic generates only one basis, the one for which the variables have the strongest link. In other words, the basis corresponding to the pair of variables which appears in most constraints.

Example 5.3.1 – Given the following constraints:

$$\begin{cases} v_1 + v_2 + v_1 \times v_2 & \leq & 3 \\ \cos(v_1) + v_3 & \leq & 10 \\ v_1 \times v_3 & \geq & 1 \end{cases}$$

The basis $B_\alpha^{1,3}$ is generated as variables v_1 and v_3 appear together in two constraints while the pair $\{v_1, v_2\}$ appears in only one constraint.

The idea of the StrongestLink heuristic is to emphasize the strongest relation between two variables. We want the new variables to be the most constrained as possible so that the solving process will have more chances to reduce their domains.

The last heuristic rely on the notion of promising scheme.

Definition 5.3.8 (Promising Scheme)**.** Let $(v_1 \dots v_n)$ be the variables set. We call *promising scheme* expressions corresponding to the following patterns: $\pm v_i \pm v_j$ or $\pm v_i \times v_j$ for $i,, j \in [\![1, n]\!]$.

These schemas are said promising as they can easily simplified in the corresponding rotated basis.

Example 5.3.2 – Let v_1 and v_2 be two variables. Consider the expression $v_1 + v_2$, the corresponding expression in the $(1, 2)$-rotated basis is:

$$\begin{aligned} v_1 + v_2 &= av_1^{1,2} - av_2^{1,2} + av_1^{1,2} + av_2^{1,2} \\ &= 2av_1^{1,2} \end{aligned}$$

where $a = \cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$. Similarly, consider the expression $v_1 \times v_2$. The corresponding expression in the $(1, 2)$-rotated basis is:

$$\begin{aligned} v_1 \times v_2 &= \left(av_1^{1,2} - av_2^{1,2}\right) \times \left(av_1^{1,2} + av_2^{1,2}\right) \\ &= \left(av_1^{1,2}\right)^2 - \left(av_2^{1,2}\right)^2 \end{aligned}$$

Definition 5.3.9 (Promising Heuristic (P))**.** Consider a CSP on variables $(v_1 \dots v_n)$ to octogonalize. This heuristic generates the basis $B_\alpha^{i,j}$ maximizing the number of promising patterns in the constraints.

Example 5.3.3 – Given the following constraints:

$$\begin{cases} v_1 + v_2 + v_1 \times v_2 & \leq & 3 \\ \cos(v_1) + v_3 & \leq & 10 \\ v_1 \times v_3 & \geq & 1 \end{cases}$$

The basis $B_\alpha^{1,2}$ is generated as there are two promising patterns with variables v_1 and v_2 ($v_1 + v_2$ and $v_1 \times v_2$) while for the pair $\{v_1, v_3\}$ there is only one promising pattern ($v_1 \times v_3$).

With this heuristic, we want to reduce the number of multiple occurrences of the new variables in the rotated constraints. We know that implementations of the Hull-consistency is sensitive to multiple occurrences of variables. By reducing the number of multiple occurrences, we hope to have a more efficient consistency and thus, to have a faster solving process.

5.4. Experimental Results

This section compares the results obtained with the octagonal solver to those obtained with a traditional interval solver on classical benchmarks.

5.4.1. *Implementation*

We have implemented a prototype of the octagonal solver, with Ibex, a C++ library for continuous constraints [CHA 09a]. We use the Ibex implementation of *HC4-Revise* [BEN 99] to propagate the constraints. The octagons are implemented with their matrix representation (DBM). Additional rotated variables and constraints are posted and dealt with as explained above.

An important point is the rotation of the constraints. The HC4 algorithm is sensitive to multiple occurrences of the variables, and the symbolic rewriting defined Section 5.1 creates multiple occurrences. Thus, the HC4 propagation on the rotated constraints could be very poor if performed directly on the rotated constraints. It is necessary to simplify the rotated constraints with respect to the number of multiple occurrences for the variables. We use the `Simplify` function of Mathematica to do this. The computation time indicated below does not include the time for this treatment, however, it is negligible compared to the solving times.

The propagator is an input of our method: we used a standard one (HC4). Further work include adding propagators such as [ARA 10], which better takes into account the symbolic expression of the constraint to improve the propagation.

5.4.2. *Methodology*

We have tested the prototype octagonal solver on problems from the Coconut benchmark [1]. These problems have been chosen depending on the type of the constraints (inequations, equations, or both) and on the number of variables.

First, we have compared the computation time to find the first solution or all the solutions with the intervals and the octagons. For these experiments, we have fixed the

1. This benchmark can be found at `http://www.mat.univie.ac.at/~neum/glopt/coconut/`

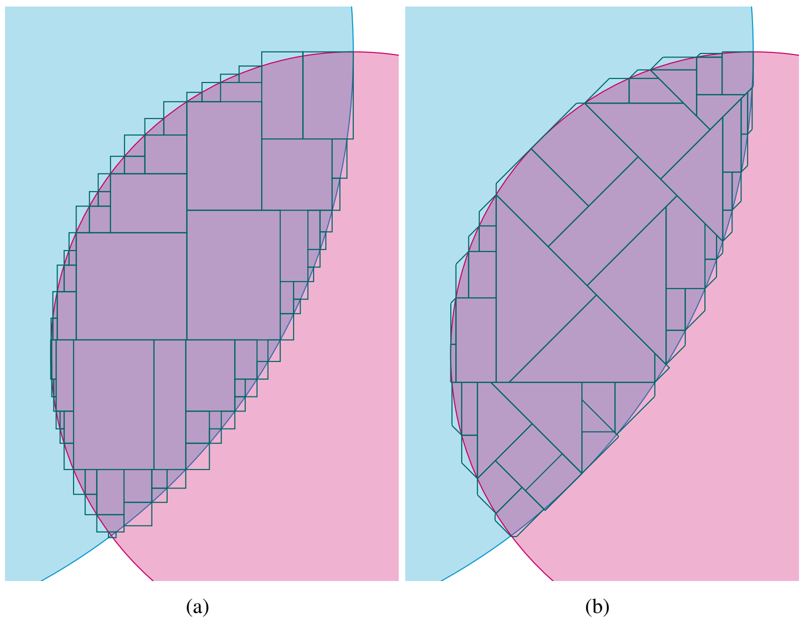


Figure 5.4. *Comparison of the results obtained with the usual interval resolution 5.4(a) to the results obtained with the octagonal resolution 5.4(b), given an intersection problem and a time limit of 10ms.*

precision parameter r to 0.01. For all the other experiments, the precision parameter r is set to 0.001. The second experiment, concerns the study of the different heuristics for the choice of variable presented Section 5.3.1. The last series of tests, compares the results obtained by the partial octagons generated by the different octagonalization heuristics (Section 5.3.2) to the one obtained with octagons and intervals.

All experiments were performed with Ibex 1.18 on a MacBook Pro Intel Core 2 Duo 2.53 GHz. All were performed with the same configurations in IBEX. In particular, the same propagators were used in order to more accurately compare the results obtained with the octagonal resolution to those obtained with the intervals. In addition, we have set a time limit to 3 hours in all cases.

5.4.3. *Results*

Figure 5.4 compares the results obtained with the interval resolution (5.4(a)) to the results obtained with the octagonal solver (5.4(b)) given a time limit of 10ms. One can see that with the standard interval solver, there is a staircase effect caused by the boxes

			Première solution		Toutes les solutions	
nom	# var	ctr type	$\mathbb{B}$	$\mathbb{O}$	$\mathbb{B}$	$\mathbb{O}$
h75	5	$\leq$	41.40	**0.03**	-	-
hs64	3	$\leq$	**0.01**	0.05	-	-
h84	5	$\leq$	5.47	**2.54**	-	**7238.74**
KinematicPair	2	$\leq$	**0.00**	**0.00**	53.09	**16.56**
pramanik	3	$=$	28.84	**0.16**	**193.14**	543.46
trigo1	10	$=$	18.93	**1.38**	**20.27**	28.84
brent-10	10	$=$	6.96	**0.54**	**17.72**	105.02
h74	4	$=$ $\leq$	305.98	**13.70**	1304.23	**566.31**
fredtest	6	$=$ $\leq$	3146.44	**19.33**	-	-

Table 5.1. *Results on problems from the Coconut benchmark. For each problem, we give the CPU time in second to find the first solution or all the solutions with the intervals ($\mathbb{B}$) and the octagons ($\mathbb{O}$).*

on the limit of the solution set. This effect is less present with the octagonal solver. Thanks to their shape, the octagons can, in some cases, be closer to the solution set.

Tables 5.1, 5.2, 5.3 respectively compare for each chosen problem the results obtained with the intervals to those obtained with the octagons. The results are compared on three criteria, the CPU computation time in seconds, le number of created nodes and the number of nodes solution. Table 5.4 compares for each selected problem the results obtained in terms of time for the different variable heuristics. Table 5.5 compares for each selected problem the computation time (in seconds) obtained by the intervals to those obtained by the octagons and the partial octagons. Table 5.6 compares for each selected problem the number of boxes created during the interval solving process to the number of octagons created during the octagonal solving. Finally, Table 5.7 compares for each selected problem the number of boxes solution to the number of octagons solution.

In all the tables, the first three columns give the name of the problem, the number of variable and the type of the constraints. Also, the dash symbol '-' stands for 'time out' (3 hours).

5.4.4. *Analysis*

We analyse here the results obtained with the different experiments.

Comparison one solution vs. all the solutions

Table 5.1 compares the results in term of CPU time needed to find the first solution or all the solutions of a problem using either the intervals or the octagons. Finding the first solution is often faster using the octagons than with the intervals. Indeed, the

			Première solution		Toutes les solutions	
nom	# var	ctr type	$\mathbb{B}$	$\mathbb{O}$	$\mathbb{B}$	$\mathbb{O}$
h75	5	$\leq$	1 024 085	**149**	-	-
hs64	3	$\leq$	217	**67**	-	-
h84	5	$\leq$	87 061	**1 407**	-	**22 066 421**
KinematicPair	2	$\leq$	45	**23**	893 083	**79 125**
pramanik	3	$=$	321 497	**457**	2 112 801	**1 551 157**
trigo1	10	$=$	10 667	**397**	11 137	**5 643**
brent-10	10	$=$	115 949	**157**	238 777	**100 049**
h74	4	$=\ \leq$	8 069 309	**138 683**	20 061 357	**1 926 455**
fredtest	6	$=\ \leq$	29 206 815	**3 281**	-	-

Table 5.2. *Results on problems from the Coconut benchmark. For each problem, we give the number of created nodes to find the first solution or all the solutions with the intervals ($\mathbb{B}$) and the octagons ($\mathbb{O}$).*

nom	# var	ctr type	$\mathbb{B}$	$\mathbb{O}$
h75	5	$\leq$	-	-
hs64	3	$\leq$	-	-
h84	5	$\leq$	-	**10 214 322**
KinematicPair	2	$\leq$	424 548	**39 555**
pramanik	3	$=$	**145 663**	210 371
trigo1	10	$=$	**12**	347
brent-10	10	$=$	854	**142**
h74	4	$=\ \leq$	700 669	**183 510**
fredtest	6	$=\ \leq$	-	-

Table 5.3. *Results on problems from the Coconut benchmark. For each problem, we give the number of solution nodes to find the first solution or all the solutions with the intervals ($\mathbb{B}$) and the octagons ($\mathbb{O}$).*

nom	# var	ctr type	LF	LOF	LCF	OS
brent-10	10	$=$	-	-	362.77	**337.48**
o32	5	$\leq$	104.62	122.82	**40.60**	40.74
ipp	8	$=$	5 105.17	5 787.5	282.319	**279.36**
trigo1	10	$=$	-	-	256.71	**253.53**
KinematicPair	2	$\leq$	**59.72**	60.74	62.91	60.78
nbody5.1	6	$=$	105.93	121.33	27.33	**27.08**
pramanik	3	$=$	396.23	414.76	**240.93**	240.96
h74	4	$=\ \leq$	2896.58	4036.89	1553.67	**647.76**

Table 5.4. *Comparison of the different variable heuristics on problems from the Coconut benchmark. For each problem, we give the CPU time in seconds to solve it using the LargestFirst heuristic (LF), the LargestOctFirst (LOF), the LargestCanFirst (LCF) ant the Oct-Split (OS).*

nom	# var	ctr type	$\mathbb{B}$	$\mathbb{O}$	CB	R	SL	P
brent-10	10	$=$	**21.58**	330.73	89.78	92.59	105.91	109.74
o32	5	$\leq$	27.25	40.74	40.74	**17.63**	20.68	21.23
ipp	8	$=$	38.83	279.36	279.36	30.14	**29.07**	41.60
trigo1	10	$=$	40.23	253.53	253.53	38.60	**37.03**	**37.03**
KinematicPair	2	$\leq$	**59.04**	60.78	60.78	60.78	60.78	60.78
nbody5.1	6	$=$	95.99	27.08	**22.13**	443.07	50.87	439.69
bellido	9	$=$	**111.12**	-	-	362.69	361.56	318.09
pramanik	3	$=$	281.80	240.96	240.96	141.94	137.42	**131.30**
caprasse	4	$=$	9175.36	-	-	**1085.21**	1131.33	2353.58
h74	4	$=$ $\leq$	-	647.76	647.76	-	**0.15**	**0.15**

Table 5.5. *Results on problems from the Coconut benchmark. For each problem, we give the CPU time in seconds to solve it using the intervals ($\mathbb{B}$), the octagons ($\mathbb{O}$) and all the partial octagons, Constraint Based (CB), Random (R), Strongest Link (SL) and Promising (P).*

nom	# var	ctr type	$\mathbb{B}$	$\mathbb{O}$	CB	R	SL	P
brent-10	10	$=$	211 885	**5 467**	5 841	175 771	205 485	21 495
o32	5	$\leq$	161 549	**25 319**	**25 319**	52 071	62 911	72 565
ipp	8	$=$	237 445	**21 963**	**21 963**	51 379	50 417	75 285
trigo1	10	$=$	13 621	**4 425**	**4 425**	6 393	5 943	5 943
KinematicPair	2	$\leq$	847 643	**373 449**	**373 449**	**373 449**	**373 449**	**373 449**
nbody5.1	6	$=$	598 521	5 435	**5 429**	578 289	137 047	542 263
bellido	9	$=$	774 333	-	-	577 367	573 481	**496 729**
pramanik	3	$=$	1 992 743	**243 951**	**243 951**	346 633	315 861	319 037
caprasse	4	$=$	150 519 891	-	-	**4 445 655**	4 472 839	9 933 597
h74	4	$=$ $\leq$	-	418 867	418 867	-	**625**	**625**

Table 5.6. *Results on problems from the Coconut benchmark. For each problem, we give the number of box created while solving with the intervals ($\mathbb{B}$), as well as the number of octagons created while solving with the octagons ($\mathbb{O}$) and all the partial octagons, Constraint Based (CB), Random (R), Strongest Link (SL) and Promising (P).*

nom	# var	ctr type	$\mathbb{B}$	$\mathbb{O}$	CB	R	SL	P
brent-10	10	$=$	825	**149**	153	636	1 643	1 765
o32	5	$\leq$	74 264	**12 523**	**12 523**	35 868	31 216	25 833
ipp	8	$=$	2 301	**2243**	**2243**	4 329	6 922	6 194
trigo1	10	$=$	40	**32**	**32**	168	140	140
KinematicPair	2	$\leq$	346 590	**186 717**	**186 717**	**186 717**	**186 717**	**186 717**
nbody5.1	6	$=$	1 012	1 003	**996**	1 794	50 526	2 230
bellido	9	$=$	**7 372**	-	-	34 756	32 482	37 508
pramanik	3	$=$	149 011	**54 659**	**54 659**	72 052	69 621	71 239
caprasse	4	$=$	1 544	-	-	164	148	**110**
h74	4	$=$ $\leq$	-	209 406	209 406	-	**293**	**293**

Table 5.7. *Results on problems from the Coconut benchmark. For each problem, we give the number of box in the computed solution by the intervals ($\mathbb{B}$), as well as the number of octagons in the solution computed with the octagons ($\mathbb{O}$) and all the partial octagons, Constraint Based (CB), Random (R), Strongest Link (SL) and Promising (P).*

octagons being more precise, they are usually closer to the solutions than the boxes, and therefore, finding the first solution is faster. However, finding all the solutions can be longer with the octagons than with the boxes. A first explanation is that constraints in problems `brent-10`, `pramanik` and `trigo1` contain many multiple occurrences of variable, which greatly increases the number of multiple occurrences of rotated variables, despite the call to the `Simplify` function in Mathematica.

Looking now to the number of created nodes during the resolution (Table 5.2), the number of octagons created is often smaller then the number of boxes created during the resolution. This comes from the fact that octagons are more accurate, and thus more time is spend during the consistency but we need to split less.

As for the last table containing results for this comparison, Table 5.3, it compares the number of solution boxes to the number of solution octagons. We can see that for two problems `pramanik` and `trigo1`, the number of octagons solution is larger than the number of solution boxes. For these two problems, octagons have trouble with the consistency for the elements on the limit of the solution set.

Comparison of heuristics for the choice of variable

Table 5.4 compares the results, in terms of CPU time, of the different variable heuristics. We can see that splitting the largest size domain (LF) is not a good strategy. Indeed, if the domains are not well propagated, this may signify that the basis in which theses domains are, is of little interest for the problem we try to solve. Restricting the choice to variables living in rotated basis (LOF) does not improve the results and even deteriorates them. This shows that there exists basis in which the domains are large and do not add relevant information. Moreover, this means that sometimes it is necessary to split in the canonical basis.

When the choice of variables is restricted to the variables in the canonical basis (LCF), the results are much better and the time limit is not exceeded for any of the selected problems. This can be explained by the fact that all the rotated bases depend on the canonical basis. And therefore, any change in the canonical basis is passed on several other basis and may improve the approximation.

Finally, the strategy Oct-Split (OS) is more often the best. By splitting in a basis of interest, this heuristic efficiently explores the search space. In addition, it suits more to the definition of the precision function, which is part of the termination criterion. For these reasons, we used this heuristic for the octagonal solving in the following experiments.

Comparison of octagonalization heuristics

By looking at the computation time Table 5.5, octagons and partial octagons obtain very poor results on problems `brent-10` and `bellido`. In these two problems, the rotated bases have very few rotated constraints. This low number of rotated constraints does not help reduce the domains of the rotated variables, and thus, does not help reduce the domains of the initial variables. As, the rotated variables are linked to the initial variables, reducing the first, reduces the latter and conversely. Conversely, if the rotated variables are not reduced, no information will be given to the initial variables. In other words, the information gain provided by these rotated constraints and variables do not compensate for the time spent to propagate them.

However for problems `h74`, `caprasse` and `nbody5.1` the octagonal resolution or one of the ones using partial octagons, greatly improves the results. Given the form of the constraints, we were hoping to improve the computation time on these problems. Problem `h74` contains constraints of the form $v_i - v_j \leq c$, for only one pair (i, j), which correspond to octagonal constraints. Thus for the (i, j)-rotated basis, these constraints are well propagated and solutions are quickly found. Note that, for this problem, when all the bases are generated, the computation time is large. This shows that the resolution spend a lot of time trying to reduce domains of variables living in bases of little interest, which do not add information. The situation is the same for problem `caprasse`. On the contrary, in problem `nbody5.1`, all the variables can be grouped in pairs following the promising pattern (Definition 5.3.8). Therefore, generating all the bases ($\mathbb{O}$) or just a subset corresponding to variables pairs (CB) greatly improves the results. Moreover, we can see that one of the pair is more important than the others, because the results obtained with the StrongestLink (SL) heuristic are not deteriorated. We can also say that this pair does not correspond to the pair with the biggest number of promising patterns, as the Promising (P) heuristic obtain very poor results. Finally, it seems that it corresponds to the basis maximizing the number of rotated constraints.

For problem `KinematicPair`, the time obtained with the octagons is equivalent to the one obtained with the intervals. For the other problems, the time obtained by one of the solving methods using partial octagons slightly improve the computation time obtained with the intervals.

Still looking at the computation time, but this time only for the results obtained for each of the octagonalization heuristics as a whole rather that on a case to case basis for each problem. Overall, we can see that, contrary to what was expected, it is the StrongestLink (SL) octagonalization heuristic which gives the best results. We expected to have better results with the Promising (P) heuristic. We can therefore deduce that the more rotated constraints is, the better the propagation of the rotated variables is. This support the hypothesis given to explain the poor results on problems `brent-10` and `bellido`.

In Table 5.6, we can see that the number of octagons created during the resolution process is always smaller that the number of boxes created. Moreover, in most of the problems, the more basis are generated for the octagons, the less octagons are created during the resolution. It follows that, the more precise the representation is, the more time is spent on the propagation and fewer splits are needed.

The conclusion is the same for Table 5.6. The number of octagons solution is usually smaller when the octagons are composed of a large number of bases.

5.5. Conclusion

In this chapter, we have implemented the unified solving method introduced in Chapter 3 with the octagon abstract domain (Chapter 4). We gave possible algorithms for the different components of the octagonal solving process, such as the consistency and propagation scheme. Moreover, we proposed different heuristic for the choice of the domain to split and the creation of partial octagons. The details of a solver based on Ibex are given. And preliminary results on a classical benchmark for continuous problems are given. These results are encouraging and show that an octagonal splitting operator and exploration strategy are mandatory to better take into account the relations within an octagon and obtain a better octagonal solving method. Several perspectives are possible such as the use of recent propagators less sensitive of multiple occurrences of variables like Mohc [ARA 10]; the definition of octagonalization heuristic offering a better compromise between StrongestLink (SL) and Constraint-Based (CB); the definition of other abstract domains in Constraint Programming such as the polyhedra [COU 78] or interval polyhedra [CHE 09].

Chapter 6

An Abstract Solver: AbSolute

In the previous chapters, we have defined some notions of Abstract Interpretation in Constraint Programming, allowing us to define a unified resolution scheme that does not depend on the chosen abstract domain. This scheme has been implemented giving us an octagonal solver. This early work makes it possible, in theory, to develop a mixed solver. However, there is a practical obstacle: the lack of representation for both integer and real variables, in solvers in Constraint Programming. A solution is to do the opposite: use Abstract Interpretation abstract domains, that do not depend on the variables type, and add the Constraint Programming resolution process in Abstract Interpretation. In this chapter, we present Constraint Programming using only notions from Abstract Interpretation. We then present an abstract solver, AbSolute, implemented on top of an abstract domain library.

Using the links highlighted in the Section 2.3.1, we express the constraints resolution in Abstract Interpretation. Solving a CSP is here seen as a concrete semantics. We thus define concrete and abstract domains, abstract operators for splits and consistency, and an iterative scheme computing an approximation of its concrete semantics. These definitions allow us to obtain an abstract solving method. We have implemented and tested this method on different problems. As the definitions named above, the details of this implementation and an analysis of the results obtained are given in this chapter.

6.1. Abstract Solving Method

Similarly to the analysis of a program in Abstract Interpretation, we define the concrete semantics of solving a constraint satisfaction problem. Then, we show that the representations used in Constraint Programming correspond to abstract domains with a size function. In addition, we show that the lower closure operators in Abstract Interpretation are similar to the propagators in Constraint Programming, and therefore local iterations are equivalent to the propagation loop. Splitting operators do not exist in Abstract Interpretation, we define them and thus obtain an abstract solving method.

6.1.1. *Concrete Solving as Concrete Semantics*

Previously, Section 2.2.1, we saw that solving a constraint satisfaction problem is similar to computing the solutions set of a conjunction of constraints. Given values for the variables, constraints answer true or false in the case of discrete variables, and true, false or maybe for continuous variables. From this observation, solving a constraint satisfaction problem can be seen as the analysis of a conjunction of tests, the constraints. The concrete semantic thus corresponds to the solutions set of the problem. We consider as concrete domain $\mathcal{D}^\flat$ the subsets of the initial search space $\hat{D} = \hat{D}_1 \times \cdots \times \hat{D}_n$ (Definition 2.2.1), that is $(\mathcal{P}(\hat{D}), \subseteq, \emptyset, \cup)$. Similarly, in Constraint Programming, each constraint C_i is associated to a propagator, which corresponds in Abstract Interpretation to a lower closure operator (Definition 2.1.8) $\rho_i^\flat : \mathcal{P}(\hat{D}) \to \mathcal{P}(\hat{D})$, such that $\rho_i^\flat(X)$ only keeps points of X satisfying constraint C_i. Finally, concrete solution of the problem is simply $S = \rho^\flat(\hat{D})$, where $\rho^\flat = \rho_1^\flat \circ \cdots \circ \rho_p^\flat$.

Furthermore, by formalizing the resolution of a constraints satisfaction problem in terms of local iterations, the solutions can be expressed as a fixpoint $\text{gfp}_{\hat{D}}\, \rho^\flat$. That is the greatest fixpoint of the composition of propagators, $\rho^\flat$, smaller than the initial search space, $\hat{D}$. Solving a problem is equivalent to computing a fixpoint. Expressing the solution set as the greatest fixpoint has already been made [SCH 05].

The solutions set being identified as the concrete domain, existing representations for domains in Constraint Programming can be seen as abstract domains.

6.1.2. *Abstract Domains Existing in Constraint Programming*

Solvers do not manipulate individual points in $\hat{D}$, but rather collections of points of certain forms, such as boxes, called domains in Constraint Programming. We now show that Constraint Programming-domains can be the base set $\mathcal{D}^\sharp$ of an abstract domain $(\mathcal{D}^\sharp, \sqsubseteq^\sharp, \perp^\sharp, \sqcup^\sharp)$ in Abstract Interpretation.

Definition 6.1.1 (Integer Cartesian Product)**.** Let $v_1, \ldots, v_n$ be variables over finite discrete domains $\hat{D}_1 \ldots \hat{D}_n$. We call integer Cartesian product any Cartesian product of integer sets and is expressed in $\mathcal{D}^\flat$ by :

$$\mathcal{S}^\sharp = \left\{ \prod_i X_i \,|\, \forall i, X_i \subseteq \hat{D}_i \right\}$$

This definition corresponds to Definition 2.2.2.

Definition 6.1.2 (Integer Box)**.** Let $v_1, \ldots, v_n$ be variables over finite discrete domains $\hat{D}_1 \ldots \hat{D}_n$. We call integer box a Cartesian product of integer intervals and is expressed in $\mathcal{D}^\flat$ by :

$$\mathcal{I}^\sharp = \left\{ \prod_i [\![a_i, b_i]\!] \,|\, \forall i,\ [\![a_i, b_i]\!] \subseteq \hat{D}_i,\ a_i \leq b_i \right\} \cup \emptyset$$

This definition corresponds to Definition 2.2.3.

Definition 6.1.3 (Box)**.** Let $v_1, \ldots, v_n$ be variables over bounded continuous domains $\hat{D}_1 \ldots \hat{D}_n$. A box is a Cartesian product of intervals and is expressed in $\mathcal{D}^\flat$ by :

$$\mathcal{B}^\sharp = \left\{ \prod_i I_i \,|\, I_i \in \mathbb{I}, I_i \subseteq \hat{D}_i \right\} \cup \emptyset$$

This definition corresponds to Definition 2.2.4.

6.1.3. *Abstract Domains Operators*

In Constraint Programming, to each representation is associated a consistency. Along the same lines, we associate to each abstract domain a consistency, and in addition to the standard operators in Abstract Interpretation, we define a monotonic *size function* $\tau : \mathcal{D}^\sharp \to \mathbb{R}^+$, used as a termination criterion for the splitting operator (Définition 6.1.7).

Example 6.1.1 – Generalized arc-consistency (Definition 2.2.7) corresponds to the abstract domain of integer Cartesian products $\mathcal{S}^\sharp$ (Definition 6.1.1), ordered by element-wise set inclusion. It is linked with the concrete domain $\mathcal{D}^\flat$ by the standard Cartesian Galois connection:

$$\begin{array}{l} \mathcal{D}^\flat \underset{\alpha_a}{\overset{\gamma_a}{\leftrightarrows}} \mathcal{S}^\sharp \\ \gamma_a(S_1, \ldots, S_n) = S_1 \times \cdots \times S_n \\ \alpha_a(X^\flat) = \lambda i.\{v \,|\, \exists (x_1, \ldots, x_n) \in X^\flat,\ x_i = v\} \end{array}$$

The size function τ_a uses the size of the largest component:

$$\tau_a(S_1, \ldots, S_n) = \max_i(|S_i|)$$

Thus, if the considered element is a solution, all variables are instantiated, so all sets are singletons and τ_a is equal to 1.

Example 6.1.2 – Bound consistency (Definition 2.2.8) corresponds to the abstract domain of integer boxes $\mathcal{I}^\sharp$ (Definition 6.1.2), ordered by element-wise interval inclusion. We have a Galois connection:

$$\begin{array}{l}
\mathcal{D}^\flat \underset{\alpha_b}{\overset{\gamma_b}{\leftrightarrows}} \mathcal{I}^\sharp \\
\gamma_b(\llbracket a_1, b_1\rrbracket, \dots, \llbracket a_n, b_n\rrbracket) = \llbracket a_1, b_1\rrbracket \times \cdots \times \llbracket a_n, b_n\rrbracket \\
\alpha_b(X^\flat) = \lambda i.\llbracket\ \min\{v \in \mathbb{Z} \mid \exists(x_1, \dots, x_n) \in X^\flat,\, x_i = v\}, \\
\qquad\qquad\qquad \max\{v \in \mathbb{Z} \mid \exists(x_1, \dots, x_n) \in X^\flat,\, x_i = v\}\rrbracket
\end{array}$$

We use as size function the length of the largest dimension plus one so that, like the integer Cartesian products, if the element is a solution, the size function is equal to 1:

$$\tau_b(\llbracket a_1, b_1\rrbracket, \dots, \llbracket a_n, b_n\rrbracket) = \max_i(b_i - a_i) + 1$$

Example 6.1.3 – Hull consistency (Definition 2.2.9) corresponds to the abstract domain of boxes with floating-point bounds $\mathcal{B}^\sharp$ (Definition 6.1.3). We use the following Galois connection:

$$\begin{array}{l}
\mathcal{D}^\flat \underset{\alpha_h}{\overset{\gamma_h}{\leftrightarrows}} \mathcal{B}^\sharp \\
\gamma_h([a_1, b_1], \dots, [a_n, b_n]) = [a_1, b_1] \times \cdots \times [a_n, b_n] \\
\alpha_h(X^\flat) = \lambda i.[\ \max\{v \in \mathbb{F} \mid \forall(x_1, \dots, x_n) \in X^\flat,\, x_i \geq v\}, \\
\qquad\qquad\qquad \min\{v \in \mathbb{F} \mid \forall(x_1, \dots, x_n) \in X^\flat,\, x_i \leq v\}]
\end{array}$$

The size function corresponds to the size of the largest dimension:

$$\tau_h([a_1, b_1], \dots, [a_n, b_n]) = \max_i(b_i - a_i)$$

We observe that to each choice corresponds a classic non-relational abstract domain in Abstract Interpretation. Moreover, it is an homogeneous Cartesian product of identical single-variable domains (representation for one variable). However, this does not need be the case, new solvers can be designed beyond the ones considered in traditional Constraint Programming by varying the abstract domains further. A first idea is to apply different consistencies to different variables which permits, in particular, mixing variables with discrete domains and variables with continuous domains. Yet, the domains still corresponds to Cartesian products. A second idea is to parametrize the solver with other abstract domains from the Abstract Interpretation literature, in particular relational or mixed domains. We choose the second idea, so the representations are no longer restricted to Cartesian products and can represent different types of variables. This idea is illustrated below.

Example 6.1.4 – The octagon abstract domain $\mathcal{O}^\sharp$ [MIN 06] assigns a (floating-point) upper bound to each binary unit expression $\pm v_i \pm v_j$ on the variables $v_1, \dots, v_n$.

$$\mathcal{O}^\sharp = \{\alpha v_i + \beta v_j \mid i, j \in \llbracket 1, n\rrbracket,\, \alpha, \beta \in \{-1, 1\}\} \to \mathbb{F}$$

It enjoys a Galois connection with $\mathcal{D}^\flat$. Let us recall here this Galois connection and the size function defined previously (Definition 4.3.2). $\forall X^\sharp \in \mathcal{O}^\sharp, \forall i, j, \in [\![1, n]\!], \forall \alpha, \beta \in \{-1, 1\}$, we denote by $X^\sharp(\alpha v_i + \beta v_j)$ the floating-point upper bound of the binary unit expression $\alpha v_i + \beta v_j$.

$$\mathcal{D}^\flat \underset{\alpha_o}{\overset{\gamma_o}{\leftrightarrows}} \mathcal{O}^\sharp$$
$$\gamma_o(X^\sharp) = \{(x_1, \ldots, x_n) \mid \forall i, j, \alpha, \beta,\ \alpha x_i + \beta x_j \leq X^\sharp(\alpha v_i + \beta v_j)\}$$
$$\alpha_o(X^\flat) = \lambda(\alpha v_i + \beta v_j).\max\{\alpha x_i + \beta x_j \mid (x_1, \ldots, x_n) \in X^\flat\}$$

$$\tau_o(X^\sharp) = \min(\ \max_{i,j,\beta} (X^\sharp(v_i + \beta v_j) + X^\sharp(-v_i - \beta v_j)),\ \max_{i} (X^\sharp(v_i + v_i) + X^\sharp(-v_i - v_i))/2\ \ )$$

Example 6.1.5 – The polyhedron domain $\mathcal{P}^\sharp$ [COU 78] abstracts sets as convex, closed polyhedra. Modern implementations [JEA 09] generally follow the "double description approach" and maintain two dual representations for each polyhedron: a set of linear constraints and a set of generators. A generator is either a vertex or a ray of the polyhedron. A ray corresponds to a vector along which, starting from any vertex of the polyhedron, any points is part of the polyhedron. However, the polyhedra we used do not have rays given that they are bounded. There is no abstraction function α for polyhedra, and so, no Galois connection. Operators are generally easier in one representation. In particular, we define the size function on generators as the maximal Euclidian distance between pairs of vertices. Let $X^\sharp \in \mathcal{P}^\sharp$,

$$\tau_p(X^\sharp) = \max_{g_i, g_j \in X^\sharp} ||g_i - g_j||$$

Example 6.1.6 – The mixed box abstract domain $\mathcal{M}^\sharp$, ordered by inclusion. Let $v_1 \ldots v_n$, the variables set and let $(v_1, \ldots, v_m), m \in [\![1, n]\!]$ the set of integer variables and $(v_{m+1}, \ldots, v_n)$ the set of real variables. The abstract domain of mixed boxes assigns to integer variables an integer interval and to real variables a real interval with floating-point bounds.

$$\mathcal{M}^\sharp = \left\{ \prod_{i=1}^{m} [\![a_i, b_i]\!] \mid \forall i,\ [\![a_i, b_i]\!] \subseteq \hat{D}_i,\ a_i \leq b_i \right\} \times \left\{ \prod_{i=m+1}^{n} I_i \mid I_i \in \mathbb{I}, I_i \subseteq \hat{D}_i \right\} \cup \emptyset$$

This abstract domain enjoys a Galois connection with $\mathcal{D}^\flat$:

$$\mathcal{D}^\flat \underset{\alpha_m}{\overset{\gamma_m}{\leftrightarrows}} \mathcal{M}^\sharp$$
$$\gamma_m([\![a_1, b_1]\!], \ldots, [\![a_m, b_m]\!], [a_{m+1}, b_{m+1}], \ldots, [a_n, b_n])$$
$$= [\![a_1, b_1]\!] \times \cdots \times [\![a_m, b_m]\!] \times [a_{m+1}, b_{m+1}] \times \cdots \times [a_n, b_n]$$
$$\alpha_m(X^\flat) = \lambda i. \begin{cases} [\![\ \min\{v \in \mathbb{Z} \mid \exists (x_1, \ldots, x_n) \in X^\flat,\ x_i = v\}, \\ \quad \max\{v \in \mathbb{Z} \mid \exists (x_1, \ldots, x_n) \in X^\flat,\ x_i = v\}\ ]\!] & i \in [\![1, m]\!] \\ [\ \max\{v \in \mathbb{F} \mid \forall (x_1, \ldots, x_n) \in X^\flat,\ x_i \geq v\}, \\ \quad \min\{v \in \mathbb{F} \mid \forall (x_1, \ldots, x_n) \in X^\flat,\ x_i \leq v\}\ \ ] & i \in [\![m+1, n]\!] \end{cases}$$

The size function corresponds to the size of the largest dimension:

$$\tau_m([\![a_1, b_1]\!], \ldots, [\![a_m, b_m]\!], [a_{m+1}, b_{m+1}], \ldots, [a_n, b_n]) = \max_i(b_i - a_i)$$

6.1.4. *Constraints and Consistency*

We now assume that an abstract domain $\mathcal{D}^\sharp$ underlying the solver is fixed.

Given the concrete semantics of the constraints $\rho^\flat = \rho^\flat_1 \circ \cdots \circ \rho^\flat_p$, and if $\mathcal{D}^\sharp$ enjoys a Galois connection $\mathcal{D}^\flat \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathcal{D}^\sharp$, then the semantics of the perfect propagator achieving the consistency for all the constraints is simply: $\alpha \circ \rho^\flat \circ \gamma$. Solvers achieve this algorithmically. They apply the propagator for each constraint in turn until a fixpoint is reached or, when this process is deemed too costly, return before a fixpoint is reached.

Remark 6.1.1 – By observing that each propagator corresponds to an abstract test transfer function $\rho^\sharp_i$ in $\mathcal{D}^\sharp$, we retrieve the local iterations proposed by Granger to analyze conjunctions of tests [GRA 92]. A trivial narrowing is used here: stop refining after an iteration limit is reached.

Additionally, each $\rho^\sharp_i$ can be internally implemented by local iterations [GRA 92], a technique which is used in both the Abstract Interpretation and Constraint Programming communities. A striking connection is the analysis in non-relation domains using forward-backward iterations on expression trees [MIN 04, §2.4.4], which is extremely similar to the HC4-revise algorithm [BEN 99] developed independently for Constraint Programming.

We can thus see the consistency as an abstraction of the concrete semantics of constraints.

6.1.5. *Disjunctive Completion and Split*

To approximate the solutions to an arbitrary precision, solvers use a coverage of finitely many abstract elements from $\mathcal{D}^\sharp$. This corresponds in Abstract Interpretation to the notion of disjunctive completion [COU 92a].

Definition 6.1.4 (Disjunctive Completion)**.** Let $\mathcal{D}^\sharp$ a set. A *disjunctive completion* $\mathcal{E}^\sharp = \mathcal{P}_{\text{finite}}(\mathcal{D}^\sharp)$ is a subset of $\mathcal{D}^\sharp$ which elements are not comparable. That is:

$$\mathcal{E}^\sharp = \{X^\sharp \subseteq \mathcal{D}^\sharp \,|\, \forall B^\sharp, C^\sharp \in X^\sharp, B^\sharp \subsetneq^\sharp C^\sharp\}$$

Example 6.1.7 – All the elements a the same level in a lattice form a disjunctive completion as they cannot be compare to each other. Consider the lattice of integer boxes $\mathcal{I}^\sharp$ with inclusion, the set

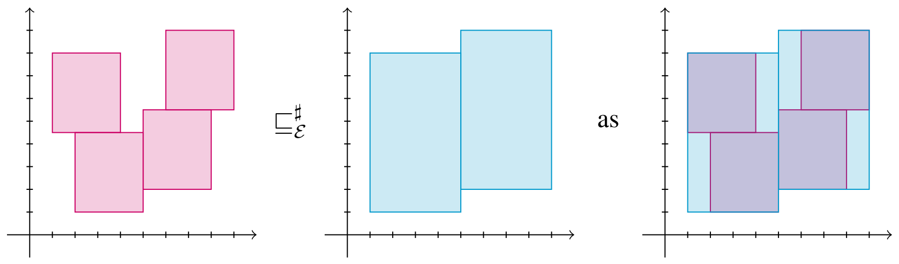
$$\{[\![1,2]\!] \times [\![1,2]\!], [\![1,2]\!] \times [\![3,7]\!]\}$$

is a disjunctive completion. Indeed, the integer box $[\![1,2]\!] \times [\![1,2]\!]$ is not comparable with the inclusion to the integer box $[\![1,2]\!] \times [\![3,7]\!]$. The first box is not included in the second and conversely.

We now consider the abstract domain $\mathcal{E}^\sharp = \mathcal{P}_{\text{finite}}(\mathcal{D}^\sharp)$, and equip it with the Smyth order $\sqsubseteq^\sharp_\mathcal{E}$, such that $\forall X^\sharp, Y^\sharp \in \mathcal{E}^\sharp$ two disjunctive completions, $X^\sharp$ is included in $Y^\sharp$ if and only if each element of $X^\sharp$ is included in one of the elements of $Y^\sharp$. It is a classic order for disjunctive completions defined as:

$$X^\sharp \sqsubseteq^\sharp_\mathcal{E} Y^\sharp \iff \forall B^\sharp \in X^\sharp, \exists C^\sharp \in Y^\sharp, B^\sharp \sqsubseteq^\sharp C^\sharp$$

Example 6.1.8 – Consider the lattice of boxes $\mathcal{B}^\sharp$ with inclusion. Let $X^\sharp = \{[2,5] \times [1,4.5], [1,4] \times [4.5,8], [5,8] \times [2,5.5], [6,9] \times [5.5,9]\}$ and $Y^\sharp = \{[1,5] \times [1,8], [5,9] \times [2,9]\}$ two disjunctive completions. We have $X^\sharp \sqsubseteq^\sharp_\mathcal{E} Y^\sharp$. Indeed each box of $X^\sharp$ is included in a box of $Y^\sharp$. Graphically, we have the following figure. The boxes on the left (in pink) are included in the boxes on the right (in blue).



The splitting operator $\oplus$, splits an element of $\mathcal{D}^\sharp$ in two or more elements. It thus transforms an element of $\mathcal{D}^\sharp$ into an element of $\mathcal{E}^\sharp$. So, it achieves the creation of new disjunctive completions. Since the splitting operator does not exist in Abstract Interpretation, we redefine it here.

Definition 6.1.5 (Split Operator)**.** A *split operator* is an operator $\oplus : \mathcal{D}^\sharp \to \mathcal{E}^\sharp$ such that $\forall e \in \mathcal{D}^\sharp$,

1) $|\oplus(e)|$ is finite,

2) $\forall e_i \in \oplus(e)$, $e_i \sqsubseteq^\sharp e$, and

3) $\gamma(e) = \bigcup\{\gamma(e_i) \mid e_i \in \oplus(e)\}$.

This definition is the abstract domains version of Definition 3.2.3.

Each element of $\oplus(e)$ is included in e (Condition 2), we have $\oplus(e) \sqsubseteq^\sharp_\mathcal{E} \{e\}$. Furthermore, Condition 3 shows that $\oplus$ is an abstraction of the identity. Thus, $\oplus$ can be freely applied at any place during the solving process without altering the soundness. The integer instantiation and the split on boxes can be retrieved with this definition.

Example 6.1.9 – The instantiation of a variable v_i in a discrete domain $X^\sharp = (S_1, \ldots, S_n) \in \mathcal{S}^\sharp$ is a splitting operator:

$$\oplus_a(X^\sharp) = \{(S_1, \ldots, S_{i-1}, x, S_{i+1}, \ldots, S_n) \mid x \in S_i\}$$

Example 6.1.10 – The instantiation of a variable v_i in a discrete domain $X^\sharp = (X_1, \ldots, X_n) \in \mathcal{I}^\sharp$ is a splitting operator:

$$\oplus_b(X^\sharp) = \{(X_1 \times \cdots \times X_{i-1} \times x \times X_{i+1} \times \cdots \times X_n) \mid x \in X_i\}$$

Example 6.1.11 – Cutting a box in two along a variable v_i in a continuous domain $X^\sharp = (I_1, \ldots, I_n) \in \mathcal{B}^\sharp$ is a splitting operator:

$$\oplus_h(X^\sharp) = \begin{array}{l} \{\ (I_1 \times \cdots \times I_{i-1} \times [a, h] \times I_{i+1} \times \cdots \times I_n), \\ \ \ (I_1 \times \cdots \times I_{i-1} \times [h, b] \times I_{i+1} \times \cdots \times I_n)\ \} \end{array}$$

where $I_i = [a, b]$ and $h = \overline{(a+b)/2}$ or $h = \underline{(a+b)/2}$.

Other splitting operators can be defined for non-relational or mixed abstract domains.

Example 6.1.12 – Given a binary unit expression $\alpha v_i + \beta v_j, i, j \in [\![1, n]\!], \alpha, \beta \in \{-1, 1\}$, we define the split on an octagon $X^\sharp \in \mathcal{O}^\sharp$ along this expression as:

$$\oplus_o(X^\sharp) = \{X^\sharp[(\alpha v_i + \beta v_j) \mapsto h],\ X^\sharp[(-\alpha v_i - \beta v_j) \mapsto -h]\}$$

where $h = (X^\sharp(\alpha v_i + \beta v_j) - X^\sharp(-\alpha v_i - \beta v_j))/2$, rounded in $\mathbb{F}$ in any direction.

This operator corresponds to the one presented Definition 4.3.1. Indeed, cutting along a binary unit expression $\alpha v_i + \beta v_j$ is equivalent to split the domain of variable $v_i^{i,j}$ or $v_j^{i,j}$ when $i \neq j$, and of variable v_i or v_j when $i = j$.

Example 6.1.13 – Given a polyhedron $X^\sharp \in \mathcal{P}^\sharp$ represented as a set of linear constraints, and a linear expression $\sum_i \beta_i v_i$. We define the splitting operator on a polyhedron along this expression as:

$$\oplus_p(X^\sharp) = \left\{ X^\sharp \cup \left\{ \sum_i \beta_i v_i \leq h \right\}, X^\sharp \cup \left\{ \sum_i \beta_i v_i \geq h \right\} \right\}$$

with $h = \dfrac{\left(\min\limits_{\gamma(X^\sharp)} \sum\limits_i \beta_i v_i + \max\limits_{\gamma(X^\sharp)} \sum\limits_i \beta_i v_i \right)}{2}$ can be computed by the Simplex algorithm [DAN 97].

Example 6.1.14 – Consider the mies box $X^\sharp = (X_1, \ldots, X_m, I_{m+1}, \ldots, I_n) \in \mathcal{M}^\sharp$. We define the splitting operator on a mixed box, such that either a discrete variable is instantiated, or the domain of a continuous variable is split:

$$\oplus_m(X^\sharp) = \begin{cases} \begin{array}{l} \{(X_1 \times \cdots \times X_{i-1} \times x \times X_{i+1} \times \cdots \times X_m \times I_{m+1} \times \cdots \times I_n) \\ \quad \mid x \in X_i\} \end{array} & \text{if } i \in [\![1, m]\!] \\ \begin{array}{l} \{(X_1 \times \ldots X_m \times I_{m+1} \times \cdots \times I_{i-1} \times [a, h] \times I_{i+1} \times \cdots \times I_n), \\ \ (X_1 \times \ldots X_m \times I_{m+1} \times \cdots \times I_{i-1} \times [h, b] \times I_{i+1} \times \cdots \times I_n)\ \} \end{array} & \text{if } i \in [\![m+1, n]\!] \end{cases}$$

where $\forall i \in [\![m+1, n]\!], I_i = [a, b]$ and $h = \overline{(a+b)/2}$ or $h = \underline{(a+b)/2}$.

These splitting operators are parametrized by the choice of a direction of cut (some variable or expression). For non-relational domains we can use two classic strategies from Constraint Programming: define and order and split each variable in turn, or split along a variable with maximal size or with the smallest domain ; that is $|S_i|$ for a variable represented with a set or $b - a$ for a variable represented with an interval $[a, b]$ or $[\![a, b]\!]$. These strategies lift naturally to octagons by replacing the set of variables with the (finite) set of unit binary expressions. For polyhedra, one can bisect the segment between two vertices that are the farthest apart, in order to minimize τ_p. However, even for relational domains, we can use a faster and simpler non-relational split, such as cut along the variable with the largest domain.

Definition 6.1.6 (Choice Operator)**.** A *choice operator* is an operator $\pi : \mathcal{E}^\sharp \to \mathcal{D}^\sharp$ such that $\forall X^\sharp \in \mathcal{E}^\sharp$, for $r \in \mathbb{R}^{>0}$ fixed,

1) $\pi(X^\sharp) \in X^\sharp$, and
2) $(\tau \circ \pi)(X^\sharp) > r$.

The choice operator choose an element in the disjunctive completion $X^\sharp$, bigger than a given value r.

To ensure the termination of the solver, we impose that any series of reductions, splits, and choices eventually outputs a small enough element for τ:

Definition 6.1.7 (Compatibility of τ and $\oplus$)**.** The two operators $\tau : \mathcal{D}^\sharp \to \mathcal{R}^+$ and $\oplus : \mathcal{D}^\sharp \to \mathcal{E}^\sharp$ are said compatible if and only if, for any reductive operator $\rho^\sharp : \mathcal{D}^\sharp \to \mathcal{D}^\sharp$ (*i.e.* $\forall X^\sharp \in \mathcal{D}^\sharp, \rho^\sharp(X^\sharp) \sqsubseteq^\sharp X^\sharp$) and any family of choice operators $\pi_i : \mathcal{E}^\sharp \to \mathcal{D}^\sharp$ (*i.e.* $\forall Y^\sharp \in \mathcal{E}^\sharp, \pi_i(Y^\sharp) \in Y^\sharp$), we have:

$$\forall e \in \mathcal{D}^\sharp,\ \forall r \in \mathbb{R}^{>0}, \exists K \text{ such that } \forall j \geq K,\ (\tau \circ \pi_j \circ \oplus \circ \rho^\sharp \circ \cdots \circ \pi_1 \circ \oplus \circ \rho^\sharp)(e) \leq r$$

All operators $\oplus$ and $\rho^\sharp$ being contracting, the operator $\pi \circ \oplus \circ \rho^\sharp$ is also reducing the domains. And thus for any r strictly positive, the previous definition is verified.

We can easily verified that the splitting operators previously introduced, $\oplus_a$, $\oplus_b$, $\oplus_h$, $\oplus_o$, $\oplus_p$, and $\oplus_m$, are respectively compatible with the size function, τ_a, τ_b, τ_h, τ_o, τ_p, and τ_m, respectively proposed for the $\mathcal{S}^\sharp$, $\mathcal{I}^\sharp$, $\mathcal{B}^\sharp$, $\mathcal{O}^\sharp$, $\mathcal{P}^\sharp$ and $\mathcal{M}^\sharp$ abstract domain.

Remark 6.1.2 – The search procedure can be represented as a search tree (defined Section 2.2.4). With this representation, the set of nodes at a given depth corresponds to a disjunction over-approximating the solutions set. Moreover, a series of reduction (ρ), selection (π), and split ($\oplus$) operators corresponds to a tree branch. Definition 6.1.7 states that each branch of the search tree is finite.

6.1.6. *Abstract Solving*

The abstract solving algorithm is given Algorithm 6.1. It maintains in $\mathrm{toExplore}$ and sols two disjunctions in $\mathcal{E}^\sharp$, and iterates the following steps:

1) choose an abstract element e in $\mathrm{toExplore}$ (**pop**),

2) apply the consistency ($\rho^\sharp$),

3) and either discard the result e, add it to the set of solutions sols, or split it ($\oplus$).

The solver starts with the maximal element $\top^\sharp$ of $\mathcal{D}^\sharp$, which represents $\gamma(\top^\sharp) = \hat{D}$.

This algorithm corresponds to Algorithm 3.1 in which the abstract domain E defined in Constraint Programming, is replaced by the abstract domain $\mathcal{D}^\sharp$.

The termination is ensured by the following proposition:

Proposition 6.1.1. *If τ and $\oplus$ are compatible, the algorithm 6.1 terminates.*

Proof.

```
list of abstract domains sols ← $\emptyset$                          /* stores the abstract solutions */
queue of abstract domains toExplore ← $\emptyset$    /* stores the abstract elements to explore */
abstract domain $e \in \mathcal{D}^\sharp$

push $\top^\sharp$ in toExplore        /* initialization with the abstract search space: $\gamma(\top^\sharp) = \hat{D}$ */

while toExplore $\neq \emptyset$ do
    $e$ ← pop(toExplore)
    $e$ ← $\rho^\sharp(e)$
    if $e \neq \emptyset$ then
        if $\tau(e) \leq r$ or isSol($e$) then   /* isSol(e) returns true if e contains only solutions */
            sols ← sols $\cup$ $e$
        else
            push $\oplus(e)$ in toExplore
        end if
    end if
end while
```

Algorithm 6.1: Our generic abstract solver

We show that the search tree is finite. Suppose that the search tree is infinite. Its width is finite by Definition 6.1.5, there would exist an infinite branch (König's lemma), which would contradict Definition 6.1.7.

The search tree being finite, algorithme 6.1 terminates.

□

Proposition 6.1.2. *Algorithm 6.1 is correct.*

Proof.

At each iteration, $\bigcup\{\gamma(x) \,|\, x \in \text{toExplore} \cup \text{sols}\}$ is an over-approximation of the set of solutions, because the consistency $\rho^\sharp$ is an abstraction of the concrete semantics ρ of the constraints and the splitting operator $\oplus$ is an abstraction of the identity. We note that abstract elements in sols are consistent and either contain only solutions or are smaller than r. The algorithm terminates when toExplore is empty, at which point sols over-approximates the set of solutions with consistent elements that contain only solutions or are smaller than r. To compute the exact set of solutions in the discrete case, it is sufficient to choose $r = 1$.

Thus, Algorithm 6.1 computes an over-approximation or an exact approximation in the discrete case of the solutions set.

□

The solving process presented in Algorithm 6.1 uses a queue data-structure, and splits the oldest abstract element first. More clever choosing strategies, such as splitting the largest element for τ, can be defined. As the two previous propositions do not depend on the choice operator, the algorithm remains correct and terminates for any strategy.

Similarly to local iterations in Abstract Interpretation, our solver performs decreasing abstract iterations. More precisely, $\mathrm{toExplore} \cup \mathrm{sols}$ is decreasing for $\sqsubseteq^{\sharp}_{\mathcal{E}}$ in the disjunctive completion domain $\mathcal{E}^{\sharp}$ at each iteration of the loop. Indeed, $\rho^{\sharp}$ is reductive in $\mathcal{D}^{\sharp}$, $\rho^{\sharp}(e) \sqsubseteq^{\sharp} e$, moreover, the element returned by the splitting operator is included in the starting element $\oplus(e) \sqsubseteq^{\sharp}_{\mathcal{E}} \{e\}$. However, our solver differs from classic Abstract Interpretation in two ways. Firstly, there is no splitting operator in Abstract Interpretation, new components in a disjunctive completion are generally added only at control-flow joins. For instance, when analyzing a conditional, two elements are created, one verifying the condition, the other not satisfying it. The first element is used to analyse the instructions in the block *then*, the other to analyse the *else* block. However, once leaving the conditional, the abstract union $\sqcup^{\sharp}$ of the two elements is performed, and the analysis continues with one element. Secondly, the solving iteration strategy is far more elaborated than in Abstract Interpretation. The use of a narrowing is replaced with a data-structure that maintains an ordered list of abstract elements and a splitting strategy that performs a refinement process and ensures its termination. Actually, more complex strategies than the simple one we presented here exist in the Constraint Programming literature. One example is the AC-5 algorithm [HEN 92] where, each time the domain of a variable changes, the variable decides which constraints need to be propagated. The design of efficient propagation algorithms is an active research area in Constraint Programming [SCH 01].

6.2. The AbSolute Solver

We have implemented a prototype abstract solver, from the ideas presented previously, to demonstrate the feasibility of our approach. We describe its main features and present experimental results.

6.2.1. *Implementation*

Our prototype solver, called AbSolute, is implemented in OCaml. It uses Apron, a library of numeric abstract domains intended primarily for static analysis [JEA 09]. Several abstract domains are implemented in Apron, such as intervals, octagons and polyhedra. Moreover, Apron offers an uniform API that hides abstract domains internal algorithms. Thus, a function can be generically implemented, without knowing which abstract domain is going to be used. For instance, the consistency can be generically implemented for any abstract domain, and do not need to be specifically implemented for each abstract domain.

Another important feature in Apron, is the fact that abstract domains are defined for both integer and real variables. In other words, an abstract domain can be defined for a set of variables that only contains integer or real variables, or that contains both integer and real variables.

Finally, Apron provides a language of arithmetic constraints sufficient to express many CSPs: equalities and inequalities over numeric expressions, including operators such as $+$, $-$, $\times$, $/$, $\sqrt{}$, power, modulo, and rounding to integers.

For all those reasons, we choose Apron and take advantages of its abstract domains, its uniform API, its management of integer and real variables and non linear constraints.

6.2.1.1. *Problem Modelization*

In Constraint Programming, a problem is formalized under the form of a CSP. To represent a CSP in AbSolute, the variables are stored in an environment composed of two tables, one for the integer variables, the other for the real variables. The constraints that can be expressed in Apron, are stored in a table. Since the domains of the variables is a notion that does not exist in Abstract Interpretation, they are translated into linear constraints and stored in the constraints table.

Example 6.2.1 – Consider the CSP on integer variables v_1, v_2 with domains $D_1 = D_2 = [\![1, 5]\!]$ and the real variable v_3 with domain $D_3 = [-10, 10]$, and with the constraints

C_1: $6v_1 + 4v_2 - 3v_3 = 0$
C_2: $v_1 \times v_2 \geq 3.5$

It is translate in AbSolute as:

```
let v1 = Var.of_string "$v_1$";;
let v2 = Var.of_string "$v_2$";;
let v3 = Var.of_string "$v_3$";;
let csp =
  let vars = Environment.make [|v1 ; v2|] [|v3|] in
  let doms = Parser.lincons1_of_lstring vars ["$v_1 \geq 1$";"$v_1 \leq 5$";"$v_2 \geq$
  $1$"; "$v_2 \leq 5$";"$v_3 \geq -10$";"$v_3 \leq 10$"] in
  let cons = Parser.tcons1_of_lstring vars ["$6 * v_1 - 4 * v_2 - 3 * v_3 = 0$";
  "$v_1 * v_2 \geq 3.5$"] in
(vars, doms, cons);;
```

The first three lines correspond to the creation of the variables. They are created using the function `Var.of_string` that takes a parameter, the string corresponding to the name of the variable. Then the CSP is created, the set of variables (`vars`) is created using the function `Environment.make` that takes as first argument the set of integer variables (here $\{v_1, v_2\}$) and as second argument, the set of real variables ($\{v_3\}$). The domains are created with function `Parser.lincons1_of_lstring` which transforms a table of string into a table of linear constraints on variables `vars`. Finally, the conjonction of constraints is created using the function `Parser.tcons1_of_lstring` which transforms a table of string into a table of constraints (that can be non-linear) on variables `vars`.

6.2.1.2. *Abstraction*

An abstract domain $\mathcal{D}^\sharp$ is chosen and initialized with the domains in the CSP. It is create using a "manager". Each abstract domain has its own manager. Then the transfert function of the constraints corresponding to the domains are called to initialize the abstract domain to the initial search space.

Example 6.2.2 – We continue the example 6.2.1. To create an abstract domain from the domains, one writes in AbSolute:

```
let abs = Abstract1.of_lincons_array man vars doms;;
```

where `man` corresponds to the type of abstract domain that we want to create. For instance one will use `Oct.manager_alloc()` for the octagons. Function `Abstract1.of_lincons_array` directly creates an abstract domains for variables `vars` with the table of linear constraints `doms`.

Finally, $\mathcal{D}^\sharp$ represents an abstract domain in Abstract Interpretation (such as the octagons) and a conjonction of linear constraints corresponding to the domains in Constraint Programming ($v_1 \geq 1$, $v_1 \leq 5$...).

6.2.1.3. *Consistency*

The test transfer function naturally provides propagators for the constraints. Internally, each domain implements its own algorithm to handle tests, including sophisticated methods to handle non-linear constraints. For instance, in the intervals abstract domain, the algorithm HC4-Revise is implemented to better propagate the non-linear constraints. For all the abstract domains, if no method is able to propagate non-linear constraints, they are linearized using the algorithm described in [MIN 04]. This algorithm replaces in each non-linear terms a subset of variables by the interval of possible values for these variables. We thus have a *quasi-linear* constraint, that is a constraint with intervals for coefficients. Then each interval $[a, b]$ is replaced by the value in the middle $(a + b)/2$, and the interval $[(a - b)/2, (b - a)/2]$ is added to the constants.

Example 6.2.3 – Still considering the CSP in Example 6.2.1, constraint C_2 can be transform in the following quasi-linear contraints:

$C_{2.1}$: $[\![1, 5]\!]v_2 \geq 3.5$
$C_{2.2}$: $[\![1, 5]\!]v_1 \geq 3.5$

Constraint $C_{2.1}$ corresponds to constraint C_2 in which, variable v_1 has been replaced by its domain $[\![1, 5]\!]$. Similarly, constraint $C_{2.2}$ corresponds to constraint C_2 in which variable v_2 has been replaced by its domain $[\![1, 5]\!]$. Finally, the intervals are replaced by their median value and the interval $[\![-2, 2]\!]$ is added to the constants. We obtain the following linear constraints:

$C'_{2.1}$: $3v_2 + [\![-2, 2]\!] \geq 3.5$
$C'_{2.2}$: $3v_1 + [\![-2, 2]\!] \geq 3.5$

Other linearization algorithms can be considered, such as the one proposed in [BOR 05] which in the last step, replace the intervals by either the upper or the lower bound according to the constraint and the starting interval.

In order to simulate the propagation loop, our solver performs local iterations until either a fixpoint or a maximum number of iterations is reached. This maximal number is set to three to avoid slow convergences and ensure fast solving.

Example 6.2.4 – Consider the following CSP with two real variables v_1 and v_2 taking their values in domains $D_1 = D_2 = [-4, 4]$ and with the constraints

C_1: $v_1 = v_2$
C_2: $v_1 = \frac{1}{2}v_2$

This problem has a unique solution $v_1 = v_2 = 0$, and the consistency has a very slow convergence. Indeed, by iteratively applying the propagators, the fixpoint is never reached. The propagator of constraint C_2 reduces the domain of v_2 to $[-2, 2]$, then the propagator of constraint C_1 reduces the domain of v_1 to $[-2, 2]$, then the propagator of constraint C_2 v_2 to $[-1, 1]$ and so on. The propagators reduce the domains of the variables by half at each call. The first three iterations are illustrated Figure 6.1.

Remark 6.2.1 – To avoid slow convergences, by default, AbSolute performs only 3 iterations in the consistency. In Constraint Programming solvers, such as Ibex, the consistency is stopped if the iteration has reduced the domains of less than 10%.

Remark 6.2.2 – In solvers in Constraint Programming, only constraints containing at least one variable which domain has been modified during the previous iteration are propagated. However, for simplicity, our solver propagates all the constraints at each step of the local iteration.

Example 6.2.5 – Continuing Example 6.2.2. Once the abstract domain created, the consistency can be called, it is performed in AbSolute by the following instruction:

```
let abs = consistency man abs cons max_iter;;
```

where `max_iter` is the maximal number of iterations that can be performed and which default value is 3. The type of the abstract domain `man` is mandatory so that Apron can execute the corresponding transfert functions.

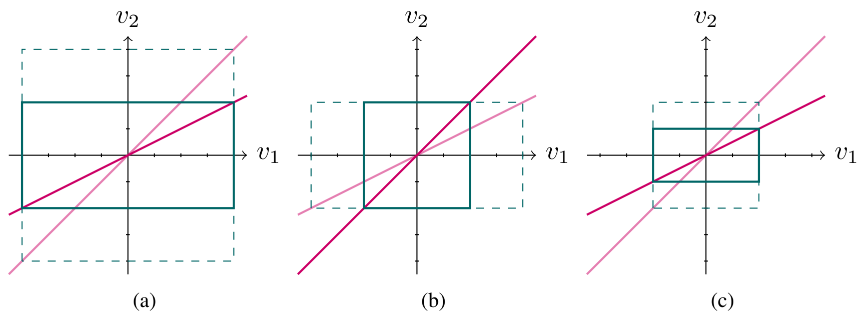
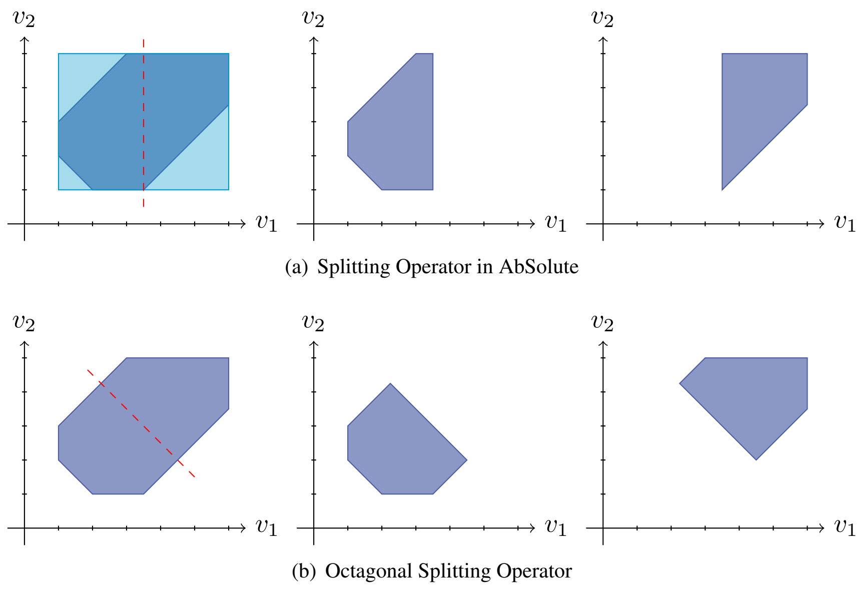


Figure 6.1. *Example of the three first iterations in the computation of the consistency with a slow convergence.*



v_2
v_1
(a) Splitting Operator in AbSolute
(b) Octagonal Splitting Operator

Figure 6.2. *Comparison between the naive splitting operator developed in AbSolute applied to an octagon, and the octagonal splitting operator given Definition 4.3.1.*

6.2.1.4. *Splitting Operator*

Currently, our solver only splits along a single variable at a time. The splitting operator splits the largest domain in two, even for relational domains and integer variables. The splitting operator uses the smallest box containing the abstract element, computes the largest dimension and cuts it in two.

Figure 6.2(a), gives an example of the splitting operator applied to an octagon. The smallest box containing the octagon is first computed. Then the variable v_1 is split along the red dashed line. For comparison, Figure 6.2(b), illustrates the split performed by the octagonal splitting operator given Definition 4.3.1.

Example 6.2.6 – Continuing Example 6.2.5. Once the abstract consistent domain computed, the splitting operator can be called, it is performed in AbSolute using the following instruction:

```
let list_abs = split man abs vars;;
```

where the splitting operator `split` returns a list of abstract domains which abstract union is equivalent to the starting abstract element `abs`.

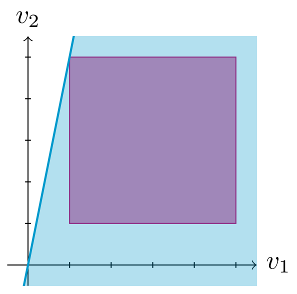
Compared to most Constraint Programming solvers, this splitting strategy is very basic, and depend neither on the abstract domain, nor on the type of variables (Figure 6.2(a)). Clever strategies from the Constraint Programming literature must be consider in future works, such as the one defined Section 2.2.5.

6.2.1.5. *Polyhedra Particular Case*

In the particular case of the polyhedron abstract domain, we realized that, in practice, the size of the polyhedron should be limited. Indeed, the solving process with polyhedron may be very slow, since the consistency is not always reductive. And can add new linear expressions that are not always relevant. For instance, consider the polyhedron on variables v_1 and v_2 composed of the following linear expressions:

$$\{v_1 \geq 1, v_1 \leq 5, v_2 \geq 1, v_2 \leq 5\}$$

The consistency for a set of constraints applied to this polyhedron may want to add to the polyhedron the linear expression $v_2 - 5v_1 \leq 0$. This linear expression is always true for any point of the polyhedron. Thus it is consider irrelevant. The figure below illustrates this example, the polyhedron is the pink square, and the new linear expression corresponds to the blue area. We can clearly see that any point of the polyhedron satisfies the linear expression since the blue area totally covers the polyhedron.



Finally, in the particular case of the polyhedron abstract domain, only the splitting operator is reductive. We thus decided to limit the size of the polyhedron by limiting its number of facets. Rather than giving a fixed number of facets, we chose to limit the value for the coefficient of the polyhedron linear expressions. By doing so, we do not need to verify after each split or consistency that the polyhedron still have the good number of facets. We only need to verify the coefficients of the newly added linear expressions.

We arbitrary set to 20 this maximum value for the coefficient of the polyhedron linear expressions. We did not have the time to do exhaustive experiments to determine, if it exists, the best possible value for this limit. Moreover, on the test we did performed, we can see that the chosen value is not limiting enough.

6.2.1.6. *Solving*

We have implemented two versions of the solving process. The first version corresponds to the solvingg method given by Algorithm 6.1. The second one, stops as soon as a first solution is found. In both versions, a queue is used to maintain the set of abstract elements to explore. No choice operator is implemented for the moment, and the abstract elements are visited following the order given by the queue. Both methods use the abstract domain and the value r as parameters. More importantly, both methods do not depend on the problem to solve, and thus do not depend on the variable type. Therefore, they can both be used to solve problems containing both integer and real variables.

6.2.2. *Experimental Results*

We have run AbSolute on two classes of problems: firstly, on continuous problems to compare AbSolute efficiency with state-of-the-art Constraint Programming solvers; secondly, on mixed problems, that these Constraint Programming solvers cannot handle while our abstract solver can.

6.2.2.1. *Continuous solving*

We use problems from the COCONUT benchmark [1], a standard CP benchmark with only real variables. We compare the results obtained with AbSolute to the ones obtained with a standard interval-based Constraint Programming continuous solver called Ibex [2]. Additionally, we compare AbSolute to our extension of Ibex to octagons (presented Section 5.4.1), which allows comparing the choice of domain (intervals versus octagons) independently from the choice of solver algorithm (classic Constraint Programming solver versus our Abstract Interpretation-based solver). Tables 6.2 and

1. Available at `http://www.mat.univie.ac.at/~neum/glopt/coconut/`.
2. Available at `http://www.emn.fr/z-info/ibex/`.

			Intervals		Octagons	
name	# vars	ctr type	Ibex	AbSolute	Ibex	AbSolute
b	4	$=$	**0.02**	0.10	0.26	**0.14**
nbody5.1	6	$=$	**95.99**	1538.25	**27.08**	$\geq$ 1h
ipp	8	$=$	**38.83**	39.24	**279.36**	817.86
brent-10	10	$=$	**21.58**	263.86	**330.73**	$\geq$ 1h
KinematicPair	2	$\leq$	59.04	**23.14**	60.78	**31.11**
biggsc4	4	$\leq$	800.91	**414.94**	1772.52	**688.56**
o32	5	$\leq$	27.36	**22.66**	40.74	**33.17**

Table 6.1. *Comparison of the CPU time in seconds to find all solutions with Ibex and AbSolute.*

			Intervals		Octagons	
name	# vars	ctr type	Ibex	AbSolute	Ibex	AbSolute
b	4	$=$	**0.009**	0.018	0.053	**0.048**
nbody5.1	6	$=$	**32.85**	708.47	**0.027**	$\geq$ 1h
ipp	8	$=$	**0.66**	9.64	19.28	**1.46**
brent-10	10	$=$	7.96	**4.57**	**0.617**	$\geq$ 1h
KinematicPair	2	$\leq$	**0.013**	0.018	0.016	**0.011**
biggsc4	4	$\leq$	**0.011**	0.022	0.096	**0.029**
o32	5	$\leq$	**0.045**	0.156	**0.021**	0.263

Table 6.2. *Comparison of the CPU time in seconds to find the first solution with Ibex and AbSolute.*

6.1 show the run time in seconds to find all the solutions or only the first solution of each problem.

On average, AbSolute is competitive with the traditional Constraint Programming approach. More precisely, it is globally slower on problems with equalities, and faster on problems with inequalities. This difference of performance does not seem to be related to the type of constraints but rather on the following ratio: the number of constraints in which a variable appears over the total number of constraints. As said previously, at each iteration, all the constraints are propagated even those for which none of their variables have changed. This increases the computation time at each step and thus increases the overall time. For instance, in the problem `brent-10`, there are ten variables, ten constraints, and each variable appears in at most three constraints. If only one variable has been modified, we will nevertheless propagate all ten constraints, instead of three at most. This may explain the timeouts observed on problems `brent-10` and `nbody5.1` with AbSolute.

Moreover, in our solver, the propagation loop is stopped after three iterations while, in the classic Constraint Programming approach, the fixpoint is reached. The

name	# vars		ctr type	$\mathcal{M}^\sharp$	$\mathcal{O}^\sharp$	$\mathcal{P}^\sharp$
	int	real				
gear4	4	2	$=$	**0.017**	0.048	0.415
st_miqp5	2	5	$\leq$	**2.636**	3.636	$\geq$ 1h
ex1263	72	20	$=$ $\leq$	**473.933**	$\geq$ 1h	$\geq$ 1h
antennes_4_3	6	2	$\leq$	**520.766**	1562.335	$\geq$ 1h

Table 6.3. *Comparison of the CPU time, in seconds, to solve mixed problems with AbSolute using different abstract domains.*

consistency in AbSolute may be less precise than the one used in Ibex. This may reduce the time spent during the propagation step but may increase the search phase.

These experimentations show that our prototype, which only features quite naive Constraint Programming strategies, behaves reasonably well on a classic benchmark. Further studies will include a deeper analysis of the performances and improvements of AbSolute on its identified weaknesses (splitting strategy, propagation loop). Since it is called at each node of the search tree, the propagation loop is a key point in complete solving method. The efficiency of a solving method depends on the efficiency of its propagation loop.

6.2.2.2. *Mixed discrete-continuous solving*

As Constraint Programming solvers seldom handle mixed problems, no standard benchmark exists. We thus gathered problems from MinLPLib,[3] a library of mixed optimisation problems from the Operational Research community. These problems are not constraints satisfaction problems, but optimization problems, with constraints to satisfy and a function to minimize. We thus needed to turn them into constraints satisfaction problems, following the approach in [BER 09]. We replaced each optimization criterion $\min f(x)$ with a constraint $|f(x) - \text{best\_known\_value}| \leq \epsilon$. We compared AbSolute to the mixed solving scheme from [BER 09], using the same ϵ and benchmarks, and found that they have similar run times (we do not provide a more detailed comparison as it would be meaningless due to the machine differences).

More interestingly, we observe that AbSolute can solve mixed problems in reasonable time and behaves better with intervals than with relational domains. A possible reason is that current propagations and heuristics are not able to fully use relational information available in octagons or polyhedra. In Section 5.3.1, we suggest that a carefully designed splitting operator is key to efficient octagons. Future work will incorporate ideas from the octagonal Constraint Programming solver into our solver and develop them further. However, Absolute is able to naturally cope with mixed

3. Available at `http://www.gamsworld.org/minlp/minlplib.htm`.

name	# vars		ctr type	$\mathcal{M}^\sharp$	$\mathcal{O}^\sharp$	$\mathcal{P}^\sharp$
	int	real				
gear4	4	2	$=$	**0.016**	0.036	0.296
st_miqp5	2	5	$\leq$	**0.672**	1.152	$\geq$ 1h
ex1263	72	20	$=\ \leq$	**8.747**	$\geq$ 1h	$\geq$ 1h
antennes_4_3	6	2	$\leq$	**3.297**	22.545	$\geq$ 1h

Table 6.4. *Comparison of the CPU time, in seconds, to find the first solution of mixed problems with AbSolute using different abstract domains.*

Constraint Programming problems in a reasonable time, opening the way to new Constraint Programming applications such as the repair and restauration of the electrical power system after a natural disaster [SIM 12]. The goal is to find a recovery plan, that is schedule and the route of each repair team in order to restore, as fast as possible, the power network after a disaster. Another application is for geometrical problems, an extension of the problem presented in [BEL 07], in these problems, the goal is to place objects of various shapes (polygons, algebraic varieties, *etc.*) so that no object overlaps another while minimizing the surface or the volume occupied.

6.3. Conclusion

In this chapter, we have explored some links between Abstract Interpretation and Constraint Programming, and used them to design a constraint satisfaction problem solving scheme built entirely on abstract domains. The preliminary results obtained with our prototype are encouraging and open the way to the development of hybrid CP–AI solvers able to naturally handle mixed constraint problems.

In future work, we wish to improve our solver by adapting and integrating advanced methods from the Constraint Programming literature. The areas of improvement include: split operators for abstract domains, specialized propagators, such as octagonal consistency or global constraints, and improvements to the propagation loop.

We built our solver on abstractions in a modular way, so that existing and new methods can be combined together, as is the case for reduced products in Abstract Interpretation. When analyzing a program, several abstract domains are generally used. In this case, the reduced product is used to communicate to abstract domains the information gathered during the analysis from other abstract domains. Ultimately, each problem should be automatically solved in the abstract domains which best fit it, as it is the case in Abstract Interpretation. As a consequence, linear problems should be solve using polyhedra and second degree polynomials with ellipsoids.

Another exciting development would be to use some methods form Constraint Programming in a static analyzer in Abstract Interpretation, such as the use of a split operator in disjunctive completion domains, and the ability of Constraint Programming to refine an abstract element to achieve completeness.

Chapter 7

Conclusion and Perspectives

7.1. Conclusion

In this book, we first studied and compared certain methods of Constraint Programming and Abstract Interpretation. We are particularly interested in tools to over-approximate impossible or difficult domains to compute exactly (concrete domain Abstract Interpretation, solutions set Constraint Programming). We relied on this study to abstract the concept of domain used Constraint Programming, in the sense that the chosen representation for domains becomes a parameter of the solver. This is equivalent to integrating abstract domains of the Abstract Interpretation framework in Constraint Programming. This allows new representations for domains to be used, including relational representations that capture some relationships between variables (for example, the polyhedron abstract domain integrates linear relationships). Furthermore, the abstract domains allowed us to define a unified solving method that do not depend on the variables types and uniformly integrate discrete and continuous solving methods of Constraint Programming.

Secondly, we have defined and implemented the weakly relational abstract domain of octagons in Constraint Programming for continuous variables. We adapted the lower closure operator existing in Abstract Interpretation on octagons to design an optimal octagonal consistency. We then used the relationship between variables expressed by octagons to guide the search. Tests performed on a classical benchmark show that these heuristics are effective. This new solving method shows the importance of developing new representations for domains in Constraint Programming, especially to integrate relational domains that are more expressive than the Cartesian product generally used in Constraint Programming. However, to do this, it is necessary for each new domain, to redefine appropriate splitting operators and consistencies, which constitutes an obstacle.

Yet, operators quite similar to the consistency have already been defined in Abstract Interpretation for many abstract domains. So, we decided to do the opposite of the previous work, and instead of integrating the concepts of Abstract Interpretation in Constraint Programming, we expressed Constraint Programming under the Abstract Interpretation framework. This is done by considering the set of all the searched solutions as the concrete domain. We have redefined constraint solving with operators in Abstract Interpretation. This allows us to use existing abstract domains directly. We implemented this method over Apron, a library of abstract domains. The prototype developed, AbSolute, that do not integrated Constraint Programming exploration strategies yet, gave promising early results.

7.2. Perspectives

This work opens up several research perspectives. First, in the short term, AbSolute must be developed to integrate more clever research techniques, such as heuristics for the variable/value choice, splitting operators or setting of the propagation loop (number and order of local iterations). For relational domains (octagons, polyhedra) in particular, work on octagons suggests that the key to effectiveness is to use existing relationships between variables to guide the search. For mixed domains, the current method can be improved by defining and implementing reduced products between reals and integer domains. Moreover, AbSolute must also be tested on real applications, especially for mixed problems such as the problem of restoration and repair of the power grid after a natural disasters [SIM 12]. The results obtained from these applications, can lead to new dedicated heuristics.

In the medium term, we should study the many existing abstract domains in Abstract Interpretation, such as the interval polyhedra, polyhedra, zonotopes or ellipsoids, as part of the Constraint Programming. This means defining associated consistencies and splitting operators. In particular, for polyhedra, we should find an effective approximation of non-linear constraints, which requires a thorough study of linearization or quasi-linearization existing algorithms.

In the long term, we finally plan to exploit the link between under-approximations in Constraint Programming and the fixed point computed in Abstract Interpretation. In Abstract Interpretation, efficient analyzers are largely based on the computation of the widening operator. We can exploit the important work on the widening to adapt it to a solving method for inner approximation in Constraint Programming.

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