On the Link between Strongly Connected Iteration Graphs and Chaotic Boolean Discrete-Time Dynamical Systems

Jacques M. Bahi[1], Jean-Francois Couchot[1], Christophe Guyeux[1], and Adrien Richard[2]

[1] Computer Science Laboratory (LIFC), University of Franche-Comté, France
{jacques.bahi,jean-francois.couchot,christophe.guyeux}@univ-fcomte.fr
[2] I3S, UMR CNRS 6070 Sophia Antipolis, France
richard@i3s.unice.fr

Abstract. Chaotic functions are characterized by sensitivity to initial conditions, transitivity, and regularity. Providing new functions with such properties is a real challenge. This work shows that one can associate with any Boolean network a continuous function, whose discrete-time iterations are chaotic if and only if the iteration graph of the Boolean network is strongly connected. Then, sufficient conditions for this strong connectivity are expressed on the interaction graph of this network, leading to a constructive method of chaotic function computation. The whole approach is evaluated in the chaos-based pseudo-random number generation context.

Keywords: Boolean network, discrete-time dynamical system, topological chaos.

1 Introduction

Chaos has attracted a lot of attention in various domains of science and engineering, *e.g.*, hash function [1], steganography [2], pseudo random number generation [3]. All these applications capitalize fundamental properties of the chaotic maps [4], namely: sensitive dependence on initial conditions, transitivity, and density of periodic points. A system is sensitive to initial conditions if any point contains, in any neighborhood, another point with a completely different future trajectory. Topological transitivity is established when, for any element, any neighborhood of its future evolution eventually overlaps with any other open set. On the contrary, a dense set of periodic points is an element of regularity that a chaotic dynamical system has to exhibit.

Chaotic discrete-time dynamical systems are iterative processes defined by a chaotic map f from a domain E to itself. Starting from any configurations $x \in E$, the system produces the sequence $x, f(x), f^2(x), f^3(x), \ldots$, where $f^k(x)$ is the k-th iterate of f at x. Works referenced above are instances of that scheme: they iterate *tent* or *logistic* maps known to be chaotic on $\mathbb{R}$.

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As far as we know, no result so far states that the chaotic properties of a function that has been theoretically proven on $\mathbb{R}$ remain valid on the floating-point numbers, which is the implementation domain. Thus, to avoid the loss of chaos this work presents an alternative: to construct, from Boolean networks $f : \mathbb{B}^n \rightarrow \mathbb{B}^n$, continuous fonctions G_f defined on the domain $[\![1;n]\!]^{\mathbb{N}} \times \mathbb{B}^n$, where $[\![1;n]\!]$ is the interval of integers $\{1, 2, \ldots, n\}$ and $\mathbb{B}$ is the Boolean domain $\{0, 1\}$. Due to the discrete nature of f, theoretical results obtained on G_f are preserved in implementation.

Furthermore, instead of finding an example of such maps and to prove the chaoticity of their discrete-time iterations, we tackle the problem of characterizing all the maps with chaotic iterations according to Devaney's chaos definition [4]. This is the first contribution. This characterization is expressed on the asynchronous iteration graph of the Boolean map f, which contains 2^n vertices. To extend the applicability of this characterization, sufficient conditions that ensure this chaoticity are expressed on the interaction graph of f, which only contains n vertices. This is the second contribution. Starting thus with an interaction graph with required properties, all the maps resulting from a Boolean network constructed on this graph have chaotic iterations. Eventually, the approach is applied on a pseudo random number generation (PRNG). Uniform distribution of the output, which is a necessary condition for PRNGs is then addressed. Functions with such property are thus characterized again on the asynchronous iteration graph. This is the third contribution. The relevance of the approach and the application to pseudo random number generation are evaluated thanks to a classical test suite.

The rest of the paper is organized as follows. Section 2 recalls discrete-time Boolean dynamical systems. Their chaoticity is characterized in Sect. 3. Sufficient conditions to obtain chaoticity are presented in Sect. 4. The application to pseudo random number generation is formalized, maps with uniform output are characterized, and PRNGs are evaluated in Sect. 5. The paper ends with a conclusion section where intended future work is presented.

2 Preliminaries

Let n be a positive integer. A Boolean network is a discrete dynamical system defined from a *Boolean map*

$$f : \mathbb{B}^n \rightarrow \mathbb{B}^n, \qquad x = (x_1, \ldots, x_n) \mapsto f(x) = (f_1(x), \ldots, f_n(x)),$$

and an *iteration scheme* (*e.g.*, parallel, sequential, asynchronous...). For instance, with the parallel iteration scheme, given an initial configuration $x^0 \in \mathbb{B}^n$, the dynamics of the system are described by the recurrence $x^{t+1} = f(x^t)$. The retained scheme only modifies one element at each iteration and is further referred by *asynchronous*. In other words, at the t^{th} iteration, only the s_t−th component is "iterated", where $s = (s_t)_{t \in \mathbb{N}}$ is a sequence of indices taken in $[\![1;n]\!]$ called "strategy". Formally, let $F_f : [\![1;n]\!] \times \mathbb{B}^n$ to $\mathbb{B}^n$ be defined by

$$F_f(i, x) = (x_1, \ldots, x_{i-1}, f_i(x), x_{i+1}, \ldots, x_n).$$

With the asynchronous iteration scheme, given an initial configuration $x^0 \in \mathbb{B}^n$ and a strategy $s \in [\![1;n]\!]^{\mathbb{N}}$, the dynamics of the network are described by the recurrence

$$x^{t+1} = F_f(s_t, x^t). \tag{1}$$

Let G_f be the map from $[\![1;n]\!]^{\mathbb{N}} \times \mathbb{B}^n$ to itself defined by

$$G_f(s,x) = (\sigma(s), F_f(s_0,x)),$$

where $\forall t \in \mathbb{N}, \sigma(s)_t = s_{t+1}$. The parallel iteration of G_f from an initial point $X^0 = (s,x^0)$ describes the "same dynamics" as the asynchronous iteration of f induced by x^0 and the strategy s (this is why G_f has been introduced).

Consider the space $\mathcal{X} = [\![1;n]\!]^{\mathbb{N}} \times \mathbb{B}^n$. The distance d between two points $X = (s,x)$ and $X' = (s',x')$ in $\mathcal{X}$ is defined by

$$d(X,X') = d_H(x,x') + d_S(s,s'), \text{ where } \begin{cases} d_H(x,x') = \displaystyle\sum_{i=1}^{n} |x_i - x'_i| \\ d_S(s,s') = \dfrac{9}{n} \displaystyle\sum_{t\in\mathbb{N}} \dfrac{|s_t - s'_t|}{10^{t+1}}. \end{cases}$$

Thus, $\lfloor d(X,X') \rfloor = d_H(x,x')$ is the Hamming distance between x and x', and $d(X,X') - \lfloor d(X,X') \rfloor = d_S(s,s')$ measures the differences between s and s'. More precisely, this floating part is less than 10^{-k} if and only if the first k terms of the two strategies are equal. Moreover, if the k^{th} digit is nonzero, then $s_k \neq s'_k$.

Let f be any map from $\mathbb{B}^n$ to itself, and $\neg : \mathbb{B}^n \to \mathbb{B}^n$ defined by $\neg(x) = (\overline{x_1}, \ldots, \overline{x_n})$. Considering this distance d on $\mathcal{X}$, it has already been proven that [5]:

- G_f is *continuous*,
- the parallel iteration of $G_\neg$ is *regular* (periodic points of $G_\neg$ are dense in $\mathcal{X}$),
- $G_\neg$ is *topologically transitive* (for all $X, Y \in \mathcal{X}$, and for all open balls B_X and B_Y centered in X and Y respectively, there exist $X' \in B_X$ and $t \in \mathbb{N}$ such that $G_\neg^t(X') \in B_Y$),
- $G_\neg$ has *sensitive dependence on initial conditions* (there exists $\delta > 0$ such that for any $X \in \mathcal{X}$ and any open ball B_X, there exist $X' \in B_X$ and $t \in \mathbb{N}$ such that $d(G_\neg^t(X), G_\neg^t(X')) > \delta$).

Particularly, $G_\neg$ is *chaotic*, according to the Devaney's definition recalled below:

Definition 1 (Devaney [4]). *A continuous map f on a metric space $(\mathcal{X}, d)$ is chaotic if it is regular, sensitive, and topologically transitive.*

In other words, quoting Devaney in [4], a chaotic dynamical system "is unpredictable because of the sensitive dependence on initial conditions. It cannot be broken down or simplified into two subsystems which do not interact because of topological transitivity. And in the midst of this random behavior, we nevertheless have an element of regularity". Let us finally remark that the definition above is redundant: Banks *et al.* have proven that sensitivity is indeed implied by regularity and transitivity [6].

3 Characterization of Chaotic Discrete-Time Dynamical Systems

In this section, we give a characterization of Boolean networks f making the iterations of any induced map G_f chaotic. This is achieved by establishing inclusion relations between the transitive, regular, and chaotic sets defined below:

- $\mathcal{T} = \{f : \mathbb{B}^n \to \mathbb{B}^n / G_f \text{ is transitive}\}$,
- $\mathcal{R} = \{f : \mathbb{B}^n \to \mathbb{B}^n / G_f \text{ is regular}\}$,
- $\mathcal{C} = \{f : \mathbb{B}^n \to \mathbb{B}^n / G_f \text{ is chaotic (Devaney)}\}$.

Let f be a map from $\mathbb{B}^n$ to itself. The *asynchronous iteration graph* associated with f is the directed graph $\Gamma(f)$ defined by: the set of vertices is $\mathbb{B}^n$; for all $x \in \mathbb{B}^n$ and $i \in [\![1;n]\!]$, the graph $\Gamma(f)$ contains an arc from x to $F_f(i,x)$. The relation between $\Gamma(f)$ and G_f is clear: there exists a path from x to x' in $\Gamma(f)$ if and only if there exists a strategy s such that the parallel iteration of G_f from the initial point (s,x) reaches the point x'. Finally, in what follows the term *iteration graph* is a shortcut for asynchronous iteration graph.

We can thus characterize $\mathcal{T}$:

Proposition 1. *G_f is transitive if and only if $\Gamma(f)$ is strongly connected.*

Proof. $\Longleftarrow$ Suppose that $\Gamma(f)$ is strongly connected. Let (s,x) and (s',x') be two points of $\mathcal{X}$, and let $\varepsilon > 0$. We will define a strategy $\tilde{s}$ such that the distance between $(\tilde{s},x)$ and (s,x) is less than ε, and such that the parallel iterations of G_f from $(\tilde{s},x)$ reaches the point (s',x').

Let $t_1 = \lfloor -\log_{10}(\varepsilon) \rfloor$, and let x'' be the configuration of $\mathbb{B}^n$ that we obtain from (s,x) after t_1 iterations of G_f. Since $\Gamma(f)$ is strongly connected, there exists a strategy s'' and $t_2 \in \mathbb{N}$ such that, x' is reached from (s'',x'') after t_2 iterations of G_f.

Now, consider the strategy $\tilde{s} = (s_0, \ldots, s_{t_1-1}, s''_0, \ldots, s''_{t_2-1}, s'_0, s'_1, s'_2, s'_3 \ldots)$. It is clear that (s',x') is reached from $(\tilde{s},x)$ after t_1+t_2 iterations of G_f, and since $\tilde{s}_t = s_t$ for $t < t_1$, by the choice of t_1, we have $d((s,x),(\tilde{s},x)) < \epsilon$. Consequently, G_f is transitive.

$\Longrightarrow$ If $\Gamma(f)$ is not strongly connected, then there exist two configurations x and x' such that $\Gamma(f)$ has no path from x to x'. Let s and s' be two strategies, and let $0 < \varepsilon < 1$. Then, for all (s'',x'') such that $d((s'',x''),(s,x)) < \varepsilon$, we have $x'' = x$, so that iteration of G_f from (s'',x'') only reaches points in $\mathcal{X}$ that are at a greater distance than one with (s',x'). So G_f is not transitive.

We now prove that:

Proposition 2. *$\mathcal{T} \subset \mathcal{R}$.*

Proof. Let $f : \mathbb{B}^n \to \mathbb{B}^n$ such that G_f is transitive (f is in $\mathcal{T}$). Let $(s,x) \in \mathcal{X}$ and $\varepsilon > 0$. To prove that f is in $\mathcal{R}$, it is sufficient to prove that there exists a strategy $\tilde{s}$ such that the distance between $(\tilde{s},x)$ and (s,x) is less than ε, and such that $(\tilde{s},x)$ is a periodic point.

Let $t_1 = \lfloor -\log_{10}(\varepsilon) \rfloor$, and let x' be the configuration that we obtain from (s, x) after t_1 iterations of G_f. According to the previous proposition, $\Gamma(f)$ is strongly connected. Thus, there exists a strategy s' and $t_2 \in \mathbb{N}$ such that x is reached from (s', x') after t_2 iterations of G_f.

Consider the strategy $\tilde{s}$ that alternates the first t_1 terms of s and the first t_2 terms of s': $\tilde{s} = (s_0, \dots, s_{t_1-1}, s'_0, \dots, s'_{t_2-1}, s_0, \dots, s_{t_1-1}, s'_0, \dots, s'_{t_2-1}, s_0, \dots)$. It is clear that $(\tilde{s}, x)$ is obtained from $(\tilde{s}, x)$ after $t_1 + t_2$ iterations of G_f. So $(\tilde{s}, x)$ is a periodic point. Since $\tilde{s}_t = s_t$ for $t < t_1$, by the choice of t_1, we have $d((s, x), (\tilde{s}, x)) < \epsilon$.

Remark 1. Inclusion of proposition 2 is strict, due to the identity map (which is regular, but not transitive).

We can thus conclude that $\mathcal{C} = \mathcal{R} \cap \mathcal{T} = \mathcal{T}$, which leads to the following characterization:

Theorem 1. *Let $f : \mathbb{B}^n \to \mathbb{B}^n$. G_f is chaotic (according to Devaney) if and only if $\Gamma(f)$ is strongly connected.*

4 Generating Strongly Connected Iteration Graph

The previous section has shown the interest of strongly connected iteration graphs. This section presents two approaches to generate functions with such property. The first is algorithmic (Sect. 4.1) whereas the second gives a sufficient condition on the interaction graph of the Boolean map f to get a strongly connected iteration graph (Sect. 4.2).

4.1 Algorithmic Generation of Strongly Connected Graphs

This section presents a first solution to compute a map f with a strongly connected graph of iterations $\Gamma(f)$. It is based on a generate and test approach.

We first consider the negation function $\neg$ whose iteration graph $\Gamma(\neg)$ is obviously strongly connected. Given a graph Γ, initialized with $\Gamma(\neg)$, the algorithm iteratively does the two following stages:

1. randomly select an edge of the current iteration graph Γ and
2. check whether the current iteration graph without that edge remains strongly connected (by a Tarjan algorithm [7], for instance). In the positive case the edge is removed from Γ,

until a rate r of removed edges is greater than a threshold given by the user. If r is close to 0% (*i.e.*, few edges are removed), there should remain about $n \times 2^n$ edges. In the opposite case, if r is close to 100%, there are about 2^n edges left. In all cases, this step returns the last graph Γ that is strongly connected. It is now then obvious to return the function f whose iteration graph is Γ.

Even if this algorithm always returns functions with stroncly connected component (SCC) iteration graph, it suffers from iteratively verifying connexity on the whole iteration graph, *i.e.*, on a graph with 2^n vertices. Next section tackles this problem: it presents sufficient conditions on a graph reduced to n elements that allow to obtain SCC iteration graph.

4.2 Sufficient Conditions to Strongly Connected Graph

We are looking for maps f such that interactions between x_i and f_j make its iteration graph $\Gamma(f)$ strongly connected. We first need additional notations and definitions. For $x \in \mathbb{B}^n$ and $i \in [\![1;n]\!]$, we denote by $\overline{x}^i$ the configuration that we obtain be switching the $i-$th component of x, that is, $\overline{x}^i = (x_1, \ldots, \overline{x_i}, \ldots, x_n)$. Information interactions between the components of the system are obtained from the *discrete Jacobian matrix* f' of f, which is defined as being the map which associates to each configuration $x \in \mathbb{B}^n$, the $n \times n$ matrix

$$f'(x) = (f_{ij}(x)), \qquad f_{ij}(x) = \frac{f_i(\overline{x}^j) - f_i(x)}{\overline{x}^j_j - x_j} \qquad (i, j \in [\![1;n]\!]).$$

More precisely, interactions are represented under the form of a signed directed graph $G(f)$ defined by: the set of vertices is $[\![1;n]\!]$, and there exists an arc from j to i of sign $s \in \{-1, 1\}$, denoted (j, s, i), if $f_{ij}(x) = s$ for at least one $x \in \mathbb{B}^n$. Note that the presence of both a positive and a negative arc from one vertex to another is allowed.

Let P be a sequence of arcs of $G(f)$ of the form

$$(i_1, s_1, i_2), (i_2, s_2, i_3), \ldots, (i_r, s_r, i_{r+1}).$$

Then, P is said to be a path of $G(f)$ of length r and of sign $\Pi_{i=1}^r s_i$, and i_{r+1} is said to be reachable from i_1. P is a *circuit* if $i_{r+1} = i_1$ and if the vertices $i_1, \ldots i_r$ are pairwise distinct. A vertex i of $G(f)$ has a positive (resp. negative) *loop*, if $G(f)$ has a positive (resp. negative) arc from i to itself.

Let $\alpha \in \mathbb{B}$. We denote by f^α the map from $\mathbb{B}^{n-1}$ to itself defined for any $x \in \mathbb{B}^{n-1}$ by

$$f^\alpha(x) = (f_1(x, \alpha), \ldots, f_{n-1}(x, \alpha)).$$

We denote by $\Gamma(f)^\alpha$ the subgraph of $\Gamma(f)$ induced by the subset $\mathbb{B}^{n-1} \times \{\alpha\}$ of $\mathbb{B}^n$. Let us give and prove the following technical lemma:

Lemma 1. *$G(f^\alpha)$ is a subgraph of $G(f)$: every arc of $G(f^\alpha)$ is an arc of $G(f)$. Furthermore, if $G(f)$ has no arc from n to a vertex $i \neq n$, then $G(f^\alpha) = G(f) \backslash n$: one obtains $G(f^\alpha)$ from $G(f)$ by removing vertex n as well as all the arcs with n as initial or final vertex.*

Proof. Suppose that $G(f^\alpha)$ has an arc from j to i of sign s. By definition, there exists $x \in \mathbb{B}^{n-1}$ such that $f^\alpha_{ij}(x) = s$, and since it is clear that $f^\alpha_{ij}(x) = f_{ij}(x, \alpha)$, we deduce that $G(f)$ has an arc from j to i of sign s. This proves the first assertion. To demonstrate the second assertion, it is sufficient to prove that if $G(f)$ has an arc from i to j of sign s, with $i, j \neq n$, then $G(f^\alpha)$ also contains this arc. So suppose that $G(f)$ has an arc from i to j of sign s, with $i, j \neq n$. Then, there exists $x \in \mathbb{B}^{n-1}$ and $\beta \in \mathbb{B}$ such that $f_{ij}(x, \beta) = s$. If $f_{ij}(x, \beta) \neq f_{ij}(x, \alpha)$, then f_i depends on the $n-$th component, in contradiction with the assumptions. So $f_{ij}(x, \alpha) = s$. It is then clear that $f^\alpha_{ij}(x) = s$, that is, $G(f^\alpha)$ has an arc from j to i of sign s.

Lemma 2. *$\Gamma(f^\alpha)$ and $\Gamma(f)^\alpha$ are isomorphic.*

Proof. Let h be the bijection from $\mathbb{B}^{n-1}$ to $\mathbb{B}^{n-1} \times \{\alpha\}$ defined by $h(x) = (x, \alpha)$ for all $x \in \mathbb{B}^{n-1}$. It is easy to see that h is an isomorphism between $\Gamma(f^\alpha)$ and $\Gamma(f)^\alpha$ that is: $\Gamma(f^\alpha)$ has an arc from x to y if and only if $\Gamma(f)^\alpha$ has an arc from $h(x)$ to $h(y)$.

Theorem 2. *Let f be a map from $\mathbb{B}^n$ to itself such that:*

1. *$G(f)$ has no cycle of length at least two;*
2. *every vertex of $G(f)$ with a positive loop has also a negative loop;*
3. *every vertex of $G(f)$ is reachable from a vertex with a negative loop.*

Then, $\Gamma(f)$ is strongly connected.

Proof. By induction on n. Let f be a map from $\mathbb{B}^n$ to itself satisfying the conditions of the statement. If $n = 1$ the result is obvious: according to the third point of the statement, $G(f)$ has a negative loop; so $f(x) = \overline{x}$ and $\Gamma(f)$ is a cycle of length two. Assume that $n > 1$ and that the theorem is valid for maps from $\mathbb{B}^{n-1}$ to itself. According to the first point of the statement, $G(f)$ contains at least one vertex i such that $G(f)$ has no arc from i to a vertex $j \neq i$. Without loss of generality, assume that n is such a vertex. Then, according to Lemma 1, f^0 and f^1 satisfy the conditions of the statement. So, by induction hypothesis, $\Gamma(f^0)$ and $\Gamma(f^1)$ are strongly connected. So, according to Lemma 2, $\Gamma(f)^0$ and $\Gamma(f)^1$ are strongly connected. To prove that $\Gamma(f)$ is strongly connected, it is sufficient to prove that $\Gamma(f)$ contains an arc $x \to y$ with $x_n = 0 < y_n$ and an arc $x \to y$ with $x_n = 1 > y_n$. In other words, it is sufficient to prove that:

$$\forall \alpha \in \mathbb{B},\ \exists x \in \mathbb{B}^n, \qquad x_n = \alpha \neq f_n(x). \tag{$*$}$$

Assume first that n has a negative loop. Then, by the definition of $G(f)$, there exists $x \in \mathbb{B}^n$ such that $f_{nn}(x) < 0$. Consequently, if $x_n = 0$, we have $f_n(x) > f_n(\overline{x}^n)$, so $x_n = 0 \neq f_n(x)$ and $\overline{x}^n_n = 1 \neq f_n(\overline{x}^n)$; and if $x_n = 1$, we have $f_n(x) < f_n(\overline{x}^n)$, so $x_n = 1 \neq f_n(x)$ and $\overline{x}^n_n = 0 \neq f_n(\overline{x}^n)$. In both cases, the condition $(*)$ holds.

Now, assume that n has no negative loop. According to the second point of the statement, n has no loop, *i.e.*, the value of $f_n(x)$ does not depend on the value of x_n. According to the third point of the statement, n is not of in-degree zero in $G(f)$, *i.e.*, f_n is not a constant. Consequently, there exists $x, y \in \mathbb{B}^n$ such that $f_n(x) = 1$ and $f_n(y) = 0$. Let $x' = (x_1, \ldots, x_{n-1}, 0)$ and $y' = (y_1, \ldots, y_{n-1}, 1)$. Since the value of $f_n(x)$ (resp. $f_n(y)$) does not depend on the value of x_n (resp. y_n), we have $f_n(x') = f_n(x) = 1 \neq x'_n$ (resp. $f_n(y') = f_n(y) = 0 \neq y'_n$). So the condition $(*)$ holds, and the theorem is proven.

Input: a function f, an iteration number b, an initial configuration x^0 (n bits)
Output: a configuration x (n bits)
$x \leftarrow x^0$;
$k \leftarrow b + \mathit{XORshift}(b+1)$;
for $i = 0, \ldots, k-1$ **do**
  $s \leftarrow \mathit{XORshift}(n)$;
  $x \leftarrow F_f(s, x)$;
end
return x;

Algorithm 1. PRNG with chaotic functions

5 Application to Pseudo Random Number Generator

This section presents a direct application of the theory developed above.

5.1 Boolean and Chaos Based PRNG

We have proposed in [8] a new family of generators that receives two PRNGs as inputs. These two generators are mixed with chaotic iterations, leading thus to a new PRNG that improves the statistical properties of each generator taken alone. Furthermore, our generator possesses various chaos properties that none of the generators used as input present. This former family of PRNGs was reduced to chaotic iterations of the negation function, *i.e.*, reduced to $G_{\neg}$. However, it is possible to use any function f such that G_f is chaotic (s.t. the graph $\Gamma(f)$ is strongly connected).

Input: the internal configuration z (a 32-bit word)
Output: y (a 32-bit word)
$z \leftarrow z \oplus (z \ll 13)$;
$z \leftarrow z \oplus (z \gg 17)$;
$z \leftarrow z \oplus (z \ll 5)$;
$y \leftarrow z$;
return y;

Algorithm 2. An arbitrary round of *XORshift* algorithm

This generator is synthesized in Algorithm 1. It takes as input: a function f; an integer b, ensuring that the number of executed iterations is at least b and at most $2b+1$; and an initial configuration x^0. It returns the new generated configuration x. Internally, it embeds two *XORshift*(k) PRNGs [9] that returns integers uniformly distributed into $[\![1; k]\!]$. *XORshift* is a category of very fast PRNGs designed by George Marsaglia, which repeatedly uses the transform of exclusive or (XOR, $\oplus$) on a number with a bit shifted version of it. This PRNG, which has a period of $2^{32} - 1 = 4.29 \times 10^9$, is summed up in Algorithm 2. It is used in our PRNG to compute the strategy length and the strategy elements.

We are then left to instantiate the function f in Algorithm 1 according to approaches detailed in Sect. 4. Next section shows how the uniformity of distribution has been taken into account.

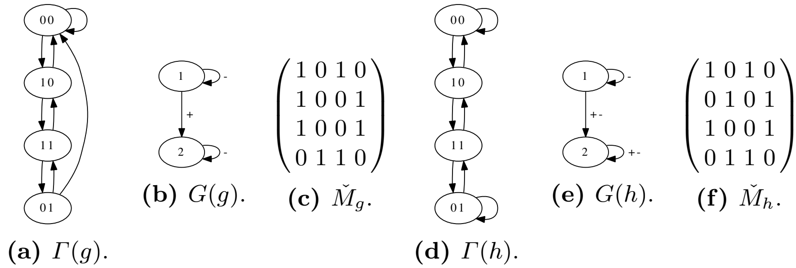


Fig. 1. Graphs of candidate functions with $n = 2$

5.2 Uniform Distribution of the Output

Let us firstly recall that a stochastic matrix is a square matrix where all entries are nonnegative and all rows sum to 1. A double stochastic matrix is a stochastic matrix where all columns sum to 1. Finally, a stochastic matrix M of size n is regular if $\exists k \in \mathbb{N}^*, \forall i, j \in [\![1; n]\!], M_{ij}^k > 0$. The following theorem is well-known:

Theorem 3. *If M is a regular stochastic matrix, then M has an unique stationary probability vector π. Moreover, if π^0 is any initial probability vector and $\pi^{k+1} = \pi^k.M$ for $k = 0, 1, \dots$ then the Markov chain π^k converges to π as k tends to infinity.*

Let us explain on a small example with 2 elements that the application of such a theorem allows to verify whether the output is uniformly distributed or not. Let then g and h be the two functions from $\mathbb{B}^2$ to itself defined in Fig. 1 and whose iteration graphs are strongly connected. As the *XORshift* PRNG is uniformly distributed, the strategy is uniform on $[\![1, 2]\!]$, and each edge of $\Gamma(g)$ and of $\Gamma(h)$ has a probability $1/2$ to be traversed. In other words, $\Gamma(g)$ is the oriented graph of a Markov chain. It is thus easy to verify that the transition matrix of such a process is $M_g = \frac{1}{2}\check{M}_g$, where $\check{M}_g$ is the adjacency matrix given in Fig. 1c, and similarly for M_h.

Both M_g and M_h are (stochastic and) regular since no element is null either in M_g^4 or in M_h^4. Furthermore, the probability vectors $\pi_g = (0.4, 0.1, 0.3, 0.2)$ and $\pi_h = (0.25, 0.25, 0.25, 0.25)$ verify $\pi_g M_g = \pi_g$ and $\pi_h M_h = \pi_h$. Thus, due to Theorem 3, for any initial probability vector π^0, we have $\lim_{k\to\infty} \pi^0 M_g^k = \pi_g$ and $\lim_{k\to\infty} \pi^0 M_h^k = \pi_h$. So the Markov process associated to h tends to the uniform distribution whereas the one associated to g does not. It induces that g shouldn't be iterated in a PRNG. On the contrary, h can be embedded into the PRNG Algorithm 1, provided the number b of iterations between two successive values is sufficiently large so that the Markov process becomes close to the uniform distribution.

Let us first prove the following technical lemma.

Lemma 3. *Let* $f : \mathbb{B}^n \to \mathbb{B}^n$*,* $\Gamma(f)$ *its iteration graph,* $\check{M}$ *the adjacency matrix of* $\Gamma(f)$*, and* M *a* $n \times n$ *matrix defined by* $M_{ij} = \frac{1}{n}\check{M}_{ij}$ *if* $i \neq j$ *and* $M_{ii} = 1 - \frac{1}{n} \sum_{j=1, j\neq i}^{n} \check{M}_{ij}$ *otherwise. Then* M *is a regular stochastic matrix iff* $\Gamma(f)$ *is strongly connected.*

Proof. Notice first that M is a stochastic matrix by construction. If there exists k s.t. $M_{ij}^k > 0$ for any $i, j \in [\![1; 2^n]\!]$, the inequality $\check{M}_{ij}^k > 0$ is thus established. Since $\check{M}_{ij}^k$ is the number of paths from i to j of length k in $\Gamma(f)$ and since such a number is positive, thus $\Gamma(f)$ is strongly connected.

Conversely, if $\Gamma(f)$ is SCC, then for all vertices i and j, a path can be found to reach j from i in at most 2^n steps. There exists thus $k_{ij} \in [\![1, 2^n]\!]$ s.t. $\check{M}_{ij}^{k_{ij}} > 0$. As all the multiples $l \times k_{ij}$ of k_{ij} are such that $\check{M}_{ij}^{l\times k_{ij}} > 0$, we can conclude that, if k is the least common multiple of $\{k_{ij} / i, j \in [\![1, 2^n]\!]\}$ thus $\forall i, j \in [\![1, 2^n]\!], \check{M}_{ij}^k > 0$. So, $\check{M}$ and thus M are regular.

With such a material, we can formulate and prove the following theorem.

Theorem 4. *Let* $f : \mathbb{B}^n \to \mathbb{B}^n$*,* $\Gamma(f)$ *its iteration graph,* $\check{M}$ *its adjacency matrix and* M *a* $n \times n$ *matrix defined as in the previous lemma. If* $\Gamma(f)$ *is SCC then the output of the PRNG detailed in Algorithm 1 follows a law that tends to the uniform distribution if and only if* M *is a double stochastic matrix.*

Proof. M is a regular stochastic matrix (Lemma 3) that has a unique stationary probability vector (Theorem 3). Let π be $\left(\frac{1}{2^n}, \ldots, \frac{1}{2^n}\right)$. We have $\pi M = \pi$ iff the sum of values of each column of M is one, *i.e.*, iff M is double stochastic.

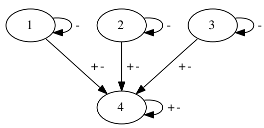
5.3 Experiments

Let us consider the interaction graph $G(f)$ given in Fig. 2a. It verifies Theorem 2: all the functions f whose interaction graph is $G(f)$ have then a strongly connected iteration graph $\Gamma(f)$. Practically, a simple constraint solving has found 520 non isomorphic functions and only 16 of them have a double stochastic matrix. Figure 2b synthesizes them by defining the images of 0,1,2,...,14,15. Let e_j be the unit vector in the canonical basis, the third column gives

$$\max_{j \in [\![1, 2^n]\!]} \{\min\{k \mid k \in \mathbb{N}, \left|\left|\pi_j M_f^k - \pi\right|\right|_2 < 10^{-4}\}\}\text{where } \pi_j \text{ is } 1/n.e_j,$$

that is the smallest iteration number that is sufficient to obtain a deviation less than 10^{-4} from the uniform distribution. Such a number is the parameter b in Algorithm 1.

Quality of produced random sequences have been evaluated with the NIST Statistical Test Suite SP 800-22 [10]. For all 15 tests of this battery, the significance level α is set to 1%: a *p-value* which is greater than 0.01 is equivalent that the keystream is accepted as random with a confidence of 99%. Synthetic results in Table. 1 show that all these functions successfully pass this statistical battery of tests.



(a) Interaction Graph

Name	Function image	b
$\mathcal{F}_1$	14, 15, 12, 13, 10, 11, 8, 9, 6, 7, 4, 5, 2, 3, 1, 0	206
$\mathcal{F}_2$	14, 15, 12, 13, 10, 11, 8, 9, 6, 7, 5, 4, 3, 2, 0, 1	94
$\mathcal{F}_3$	14, 15, 12, 13, 10, 11, 8, 9, 6, 7, 5, 4, 3, 2, 1, 0	69
$\mathcal{F}_4$	14, 15, 12, 13, 10, 11, 9, 8, 6, 7, 5, 4, 3, 2, 0, 1	56
$\mathcal{F}_5$	14, 15, 12, 13, 10, 11, 9, 8, 6, 7, 5, 4, 3, 2, 1, 0	48
$\mathcal{F}_6$	14, 15, 12, 13, 10, 11, 9, 8, 7, 6, 4, 5, 2, 3, 0, 1	86
$\mathcal{F}_7$	14, 15, 12, 13, 10, 11, 9, 8, 7, 6, 4, 5, 2, 3, 1, 0	58
$\mathcal{F}_8$	14, 15, 12, 13, 10, 11, 9, 8, 7, 6, 4, 5, 3, 2, 1, 0	46
$\mathcal{F}_9$	14, 15, 12, 13, 10, 11, 9, 8, 7, 6, 5, 4, 3, 2, 0, 1	42
$\mathcal{F}_{10}$	14, 15, 12, 13, 10, 11, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0	69
$\mathcal{F}_{11}$	14, 15, 12, 13, 11, 10, 9, 8, 7, 6, 5, 4, 2, 3, 1, 0	58
$\mathcal{F}_{12}$	14, 15, 13, 12, 11, 10, 8, 9, 7, 6, 4, 5, 2, 3, 1, 0	35
$\mathcal{F}_{13}$	14, 15, 13, 12, 11, 10, 8, 9, 7, 6, 4, 5, 3, 2, 1, 0	56
$\mathcal{F}_{14}$	14, 15, 13, 12, 11, 10, 8, 9, 7, 6, 5, 4, 3, 2, 1, 0	94
$\mathcal{F}_{15}$	14, 15, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 0, 1	86
$\mathcal{F}_{16}$	14, 15, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0	206

(b) Double Stochastic Functions

Fig. 2. Chaotic Functions Candidates with $n = 4$

Table 1. NIST Test Evaluation of PRNG instances

Property	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{F}_4$	$\mathcal{F}_5$	$\mathcal{F}_6$	$\mathcal{F}_7$	$\mathcal{F}_8$	$\mathcal{F}_9$	$\mathcal{F}_{10}$	$\mathcal{F}_{11}$	$\mathcal{F}_{12}$	$\mathcal{F}_{13}$	$\mathcal{F}_{14}$	$\mathcal{F}_{15}$	$\mathcal{F}_{16}$
Frequency	77.9	15.4	83.4	59.6	16.3	38.4	20.2	29.0	77.9	21.3	65.8	85.1	51.4	35.0	77.9	92.4
BlockFrequency	88.3	36.7	43.7	81.7	79.8	5.9	19.2	2.7	98.8	1.0	21.3	63.7	1.4	7.6	99.1	33.5
CumulativeSums	76.4	86.6	8.7	66.7	2.2	52.6	20.8	80.4	9.8	54.0	73.6	80.1	60.7	79.7	76.0	44.7
Runs	5.2	41.9	59.6	89.8	23.7	76.0	77.9	79.8	45.6	59.6	89.8	2.4	96.4	10.9	72.0	11.5
LongestRun	21.3	93.6	69.9	23.7	33.5	30.4	41.9	43.7	30.4	17.2	41.9	51.4	59.6	65.8	11.5	61.6
Rank	1.0	41.9	35.0	45.6	51.4	20.2	31.9	83.4	89.8	38.4	61.6	4.0	21.3	69.9	47.5	95.6
FFT	40.1	92.4	97.8	86.8	43.7	38.4	76.0	57.5	36.7	35.0	55.4	57.5	86.8	76.0	31.9	7.6
NonOverlappingTemplate	49.0	45.7	50.5	51.0	48.8	51.2	51.6	50.9	50.9	48.8	45.5	47.3	47.0	49.2	48.6	46.4
OverlappingTemplate	27.6	10.9	53.4	61.6	16.3	2.7	59.6	94.6	88.3	55.4	76.0	23.7	47.5	91.1	65.8	81.7
Universal	24.9	35.0	72.0	51.4	20.2	74.0	40.1	23.7	9.1	72.0	4.9	13.7	14.5	1.8	93.6	65.8
ApproximateEntropy	33.5	57.5	65.8	53.4	26.2	98.3	53.4	63.7	38.4	6.7	53.4	19.2	20.2	27.6	67.9	88.3
RandomExcursions	29.8	35.7	40.9	36.3	54.8	50.8	43.5	46.0	39.1	40.8	29.6	42.0	34.8	33.8	63.0	46.3
RandomExcursionsVariant	32.2	40.2	23.0	39.6	47.5	37.2	56.9	54.6	53.3	31.5	23.0	38.1	52.3	57.1	47.7	40.8
Serial	56.9	58.5	70.4	73.2	31.3	45.9	60.8	39.9	57.7	21.2	6.4	15.6	44.7	31.4	71.7	49.1
LinearComplexity	24.9	23.7	96.4	61.6	83.4	49.4	49.4	18.2	3.5	76.0	24.9	97.2	38.4	38.4	1.1	8.6

6 Conclusion and Future Work

This work has shown that discrete-time dynamical systems G_f are chaotic iff embedded Boolean maps f have strongly connected iteration graph $\Gamma(f)$. Sufficient conditions on its interaction graph $G(f)$ have been further proven to ensure this strong connexity. Finally, we have proven that the output of such a function is uniformly distributed iff the induced Markov chain can be represented as a double stochastic matrix. We have applied such a complete theoretical work on chaos to pseudo random number generation and all experiments have confirmed theoretical results. As far as we know, this work is the first one that allows to *compute* new functions whose chaoticity is proven and preserved during implementation. The approach relevance has been shown on PRNGs but is not limited to that domain. In fact, this whole work has applications everywhere chaoticity is a possible answer, e.g., in hash functions, digital watermarking...

In a future work, we will investigate whether the characterization of uniform distribution may be expressed in terms of interaction graph, avoiding thus to generate functions and to check later whether they induce double stochastic Markov matrix. The impact of the description of chaotic iterations as Markov processes will be studied more largely. We will look for new characterizations concerning other relevant topological properties of disorder, such as topological entropy, expansivity, Lyapunov exponent, instability, etc. Finally, the relation between these mathematical definitions and intended properties for each targeted application will be investigated too, specifically in the security field.

References

1. Yi, X.: Hash function based on chaotic tent maps. IEEE Transactions on Circuits and Systems II: Express Briefs 52(6), 354–357 (2005)
2. Dawei, Z., Guanrong, C., Wenbo, L.: A chaos-based robust wavelet-domain watermarking algorithm. Chaos, Solitons and Fractals 22, 47–54 (2004)
3. Stojanovski, T., Kocarev, L.: Chaos-based random number generators-part I: analysis [cryptography]. IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications 48(3), 281–288 (2001)
4. Devaney, R.L.: An Introduction to Chaotic Dynamical Systems, 2nd edn. Addison-Wesley, Redwood City (1989)
5. Bahi, J.M., Guyeux, C.: Topological chaos and chaotic iterations, application to hash functions. In: WCCI 2010 IEEE World Congress on Computational Intelligence, Barcelona, Spain, pp. 1–7 (2010)
6. Banks, J., Brooks, J., Cairns, G., Stacey, P.: On devaney's definition of chaos. Amer. Math. Monthly 99, 332–334 (1992)
7. Tarjan, R.: Depth-first search and linear graph algorithms. SIAM Journal on Computing 1(2), 146–160 (1972)
8. Bahi, J., Guyeux, C., Wang, Q.: A novel pseudo-random generator based on discrete chaotic iterations. In: INTERNET 2009 1-st Int. Conf. on Evolving Internet, Cannes, France, pp. 71–76 (2009)
9. Marsaglia, G.: Xorshift rngs. Journal of Statistical Software 8(14), 1–6 (2003)
10. Rukhin, A., Soto, J., Nechvatal, J., Smid, M., Barker, E., Leigh, S., Levenson, M., Vangel, M., Banks, D., Heckert, A., Dray, J., Vo, S.: A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications. National Institute of Standards and Technology (2010), http://csrc.nist.gov/groups/ST/toolkit/rng/documents/SP800-22rev1a.pdf