On the Dynamics of Bounded-Degree Automata Networks

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Abstract. Automata networks can be seen as bare finite dynamical systems, but their growing theory has shown the importance of the underlying communication graph of such networks. This paper tackles the question of what dynamics can be realized up to isomorphism if we suppose that the communication graph has bounded degree. We prove several negative results about parameters like the number of fixed point or the rank. We also give bounds on the complexity of the problem of recognizing such dynamics. However, we leave open the embarrassingly simple question of whether a dynamics consisting of a single cycle can be realized with bounded degree.

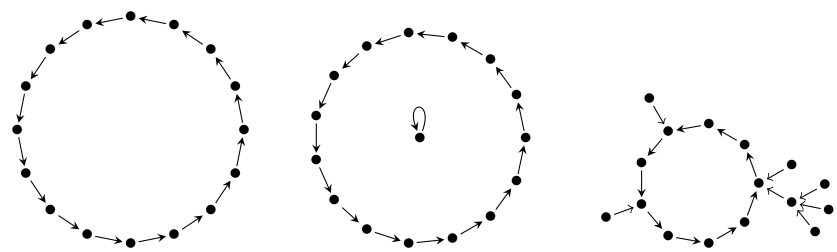
1 Introduction

One possible definition for a boolean automata network is simply a self-map $F : \{0,1\}^n \to \{0,1\}^n$. This definition forgets about the computational aspect of the model, which consists in, seen from a dual point of view, a set of n automata linked by some arcs, and each holding a bit that they can update depending on that of their incoming neighbors.

As a model of computation generalizing finite cellular automata, this communication graph is quite relevant, and it is natural to constrain it, and in particular try to restrict the possible degrees: a small degree indeed reprensents simple local computations. Note indeed that a complete communication graph can yield any dynamics $F : \{0,1\}^n \to \{0,1\}^n$.

In this paper, we address the question of how restrictions on the communication graph, and in particular bounding its degrees, can impose restrictions on the possible dynamics. For instance, in Figure 1, one can see three (families of) graphs representing possible dynamics. Which are the ones that can be realized by communication graphs with small degree?

In Section 3, we establish bounds on different parameters of the dynamics depending on the degree of communication graphs. This in particular allows to show that the family of dynamics from Figure 1(c) cannot be realized with a bounded-degree communication graph. In Section 4, we give some constructions using feedback shift registers that in particular allows to realize dynamics of



(a) a cycle of length 2^n.

(b) a cycle of length $2^n - 1$ and an isolated vertex.

(c) a cycle of length $2^n - C$ and a size-C forest plugged to it.

Fig. 1. Three examples of dynamics on 2^4 configurations.

the type from Figure 1(b) with communication graphs of degree 2. Finally, in Section 5, we give upper and lower bounds for the computational complexity of recognizing dynamics that can be realized with a bounded-degree communication graph.

However, we leave open the question about the minimum degree necessary to realize dynamics from Figure 1(a). Prior to this work, J. Aracena has communicated to us the conjecture that such dynamics requires unbounded degree. This also appears in [2] with various intermediate results.

2 Definitions and notations

Consider a finite alphabet Q with $q = |Q|$ symbols. Without loss of generality, $Q = \{0, \ldots, q-1\}$. Consider also a set $V = \{1, \ldots, n\}$ of n *nodes*. A *configuration* $x = (x_i)_{i \in N} \in Q^V$ is a function $V \to Q$. For every $U \subseteq V$, we denote $x_U : U \to Q$ the restriction of x to U, *i.e.*, $x_i = (c_U)_i$ for every $i \in X$. Given a *pattern* $u \in Q^U$, we define the *cylinder* $[u] = \{x \in Q^V : x_U = u\}$.

An *automata network* (AN) is a map $F : Q^V \to Q^V$. It can be represented as a *dynamics graph*, like those from Figure 1, by linking each configuration x to its image $F(x)$. This graph is denoted by $\mathcal{D}(F)$. A configuration x such that $F(x) = x$ is called a fixed point, and the number of fixed points of F is denoted by $\mathrm{fp}(F)$. The *rank* of F is its number of images and is denoted by $\mathrm{rk}(F)$. The set of ANs with alphabet of size q and with n nodes is denoted $\mathcal{F}(n, q)$.

A *communication graph* for F is a graph over vertex set V such that for every $i \in V$, and every $x, x' \in Q^V$ which agree over the in-neighborhood $N^{\mathrm{in}}(i) \subset V$ of i, $F(x)_{N^{\mathrm{in}}(i)} = F(x')_{N^{\mathrm{in}}(i)}$. In other words, the value $F(x)_i$ is updated thanks to a local function $f_i : Q^V \to Q$ which depends only on the values $x_{N^{\mathrm{in}}(i)}$. For $U \subseteq V$, we may also denote $f_U(x) = F(x)_U$. The *interaction graph* of F, denoted $G(F)$, is the minimal communication graph of F. Its *degree* is the maximum in-degree of a vertex in $G(F)$. We denote by $\mathcal{F}(n, q, d)$ the set ANs from $\mathcal{F}(n, q)$ which can be defined with a communication graph of degree at most d.

3 Non-local dynamics

Here we prove that some dynamics are intrinsically non-local in the sense that they cannot be realized by bounded-degree networks, even up to isomorphism.

Our first result shows that if $G(F)$ has bounded degree and F is not the identity, then the number of fixed points of F cannot be closed too close to q^n.

Proposition 1. *Let* $F \in \mathcal{F}(n,q,d)$ *with* $\mathrm{fp}(F) < q^n$*. Then* $\mathrm{fp}(F) \leq q^n - q^{n-d}$*.*

Proof. Since F is not the identity map, there exists $x \in Q^V$ such that $f_i(x) \neq x_i$ for some $i \in V$. There are two cases. If $i \notin N^{\mathrm{in}}(i)$, then for every pattern $u \in Q^{V \setminus \{i\}}$, there is a unique configuration $y \in [u]$ such that $f_i(y) = y_i = f_i(u)$; then, $\mathrm{fp}(F) \leq q^{n-1} \leq q^n - q^{n-d}$. If $i \in N^{\mathrm{in}}(i)$, then let $u = x_{N^{\mathrm{in}}(i)}$; for every configuration $y \in [u]$, $f_i(y) = f_i(x) \neq x_i = y_i$ and y is not a fixed point. Therefore, $\mathrm{fp}(F) \leq q^n - q^{n-d}$. □

Remark 1. The bound from the previous lemma is tight: indeed let $F(x) = x$ if $x_{1,\dots,d} \neq 0^d$ and $\pi x_1 x_{2,\dots,n}$ otherwise, where π is a permutation of Q without fixed point. Then F is an AN of degree $n-1$ with $q^n - q$ fixed points.

Proposition 1 can be generalised to the powers of F. First, note that if $F \in \mathcal{F}(q,n,d)$ then $F^k \in \mathcal{F}(q,n,d^k)$ for every $k \geq 1$ (because from $G(F)$ of degree $\leq d$ we obtain a communication graph for F^k by putting an edge for each path of length k). By combining this remark and Proposition 1, we obtain that, if $\mathrm{fp}(F^k) < q^n$ then $\mathrm{fp}(F^k) \leq q^n - q^{n-d^k}$.

As an application, we can easily find bijections without fixed points that force large communication degrees. Suppose for instance that the dynamics of $F \in \mathcal{F}(2,n)$ consists of $2^{n-1} - 2$ limit cycles of length 2 and one limit cycle of length 4. Then F^2 has exactly $2^n - 4$ fixed points. Denoting by d the degree of $G(F)$, we obtain that $2^n - 4 = \mathrm{fp}(F^2) \leq 2^n - 2^{n-d^2}$ and thus $d \geq \sqrt{n-2}$.

Remark 2. There are about $e^{\sqrt{2^n}}$ nonisomorphic bijective AN, but only $(q^{q^d})^n$ AN with degree $\leq d$. So few bijective AN have a realization with bounded degree.

Our second result shows that if $G(F)$ has bounded degree and F is not a bijection, then the rank of F cannot be closed too close to q^n.

Theorem 1. *Let* $F \in \mathcal{F}(n,q,d)$ *with* $rk(F) < q^n$*. Then* $rk(F) \leq q^n - \frac{n}{d+1}$*.*

In particular, the family of dynamics depicted in Figure 1(c) is impossible to realize with bounded-degree ANs. However, Theorem 1 fails among bijective ANs of fixed degree, such as the dynamics depicted in Figure 1(c), as we will see in Section 4.

The key part of the proof of Theorem 1 consists in proving that AN dynamics cannot be close to bijective without being bijective (Lemma 2). We need some definitions. We say that $F \in \mathcal{F}(n,q)$ is *k-balanced* ($k \leq n$) if for any $U \subseteq V$ with $|U| = k$ and for any pattern $u \in Q^U$, it holds $\left|F^{-1}([u])\right| = q^{n-k}$. Note that if

F is bijective then it is k-balanced for all $1 \le k \le n$. Moreover, if F is $(k+1)$-balanced, then it is k-balanced.

Given any property $\mathcal{P} \subseteq \mathcal{F}(n,q)$ of ANs, we say that a given $F \in \mathcal{F}(n,q)$ is *k-almost $\mathcal{P}$* ($k \le q^n$) if there exists $F' \in \mathcal{P}$ such that F and F' differ on at most k configurations, *i.e.* $\left|\{x \in Q^V : F'(x) \neq F(x)\}\right| \le k$. Observe that if the base property $\mathcal{P}$ is invariant under isomorphism, then being k-almost $\mathcal{P}$ is also invariant under isomorphism. For instance, *being k-almost bijective* is invariant under isomorphism.

Lemma 1. *$F \in \mathcal{F}(n,q)$ is k-almost bijective if and only if rk$(F) \ge q^n - k$.*

Proof. $\Rightarrow$: Suppose $F \in \mathcal{F}(n,q)$ is k-almost bijective. Then there exists $F' \in \mathcal{F}(n,q)$ bijective and $X \subseteq Q^V$ with $|X| = q^n - k$ and $F'(X) = F(X)$. Since F' is bijective $|F'(X)| = |X| = q^n - k$ and since $F'(X) = F(X) \subseteq F(Q^V)$ we have $\mathrm{rk}(F) \ge q^n - k$.

$\Leftarrow$: Suppose that F has rank $q^n - k$. Let $Y = \{y^1, y^2, \ldots, y^{q^n-k}\} \subseteq F(Q^V)$. Let $X = \{x^1, x^2, \ldots, x^{q^n-k}\}$ with $F(x^i) = y^i$ for all $1 \le i \le q^n - k$. Let $\overline{Y} = \{\overline{y}^1, \ldots, \overline{y}^k\} = Q^V \setminus Y$ and $\overline{X} = \{\overline{x}^1, \ldots, \overline{x}^k\} = Q^V \setminus X$. Let $F' \in \mathcal{F}(n,q)$ such that $F'(x^i) = y^i$ for all $1 \le i \le q^n - k$ and $F'(\overline{x}^i) = \overline{y}^i$ for all $1 \le i \le k$. F' is bijective and differs from F is k configurations. So F is k-almost bijective. $\square$

Lemma 2. *Let $1 \le k < n$. If $F \in \mathcal{F}(n,q)$ is k-balanced and k-almost bijective, then it is bijective.*

Proof. Suppose by contradiction that F is not bijective. If F is k-almost bijective, then there is some bijective $F' \in \mathcal{F}(n,q)$ which differs from F over a set of exactly $1 \le \ell \le k$ configurations, denoted $X = \{x_1, \ldots, x_\ell\}$. Because F' is bijective, the configurations $F'(x_1), \ldots, F'(x_\ell)$ are all distinct; moreover we cannot have $F(X) = F'(X)$ because $F(Q^V) \neq F'(Q^V)$ since one is bijective and not the other. So there are two cases:

Case 1: Assume that there exists $\le j \le \ell$ such that $F(x_j) \notin F'(X)$. In this case, the set $F'(X) \cup \{F(x_j)\}$ contains $\ell + 1$ distinct configurations. Let us inductively build a subset $U \subseteq V$ with $|U| = \ell$ such that the restrictions of these configurations to U are all distinct, *i.e.*,

$$\left|\{x_U \in Q^U : x \in F'(X)\} \cup \{F(x_j)\}\right| = \ell + 1.$$

When adding a configuration x in the set, either it is different over U to all previously included ones, then one does not need to change U; otherwise it is equal to at most one d over U, then simply add an automaton that distinguishes x from d. Then this contradicts the fact that F is ℓ-balanced because the pattern $F(x_j)_U \in Q^U$ has at least one pre-image more under F than it has under F', which must be ℓ-balanced because bijective. Indeed, the pattern $F(x_j)_U$ has no pre-image by F' in X and F is similar to F' on $Q^V \setminus X$ but $x_j \in X$ is a pre-image of $F(x_j)_U$ by F.

Case 2: Otherwise, there are $x_i \neq x_j$ such that $F(x_i) = F(x_j)$. Then, following the same idea as in the previous case, we can find $U \subseteq V$ of size ℓ such that the restrictions to U of configurations $F'(X)$ are all distinct. For such an U, the pattern $F(x_j)_U \in Q^U$ has at least one pre-image more under F than it has under F', which contradicts the fact that both F and F' are ℓ-balanced. $\square$

Lemma 3. *Consider $F \in \mathcal{F}(n,q,d)$ and $U \subseteq V$ with $k = |U| \leq \lfloor n/d \rfloor$. Then for any pattern $u \in Q^U$, the number of pre-images under F of the corresponding cylinder is a multiple of q^{n-kd}.*

Proof. Since the degree of $G(F)$ is upper-bounded by d, f_U only depends of $Y = \bigcup_{i\in U} N^{\text{in}}(i)$ and $|Y| \leq kd$. In other words, for every $x \in Q^U$ such that $f_U(x) = u$, we have $f_U([x_U]) = u$. Hence, $|F^{-1}([u])| = |\{v \in Q^Y \mid f_U(v) = u\}|q^{n-|Y|}$. Since $|Y| \leq k \cdot d$, this is a multiple of q^{n-kd}. $\square$

Combining Lemma 2 and Lemma 3 we obtain the following.

Lemma 4. *Let $F \in \mathcal{F}(n,q,d)$ and $1 \leq k \leq \lfloor n/d \rfloor$. If F is k-almost bijective but not bijective then $k \geq q^{n-dk}$.*

Proof. By Lemma 2 F cannot be k-balanced, so there is a cylinder $u \in Q^U$ with $|U| = k$ such that $\alpha = |F^{-1}([u])| > q^{n-k}$. However, Lemma 3 gives that $\alpha = mq^{n-kd}$ for some $m > 0$. We deduce that $\alpha \geq q^{n-k} + q^{n-kd}$, so at least q^{n-kd} changes in F are necessary to recover k-balance (hence bijectivity). Since F is assumed k-almost bijective, we deduce $k \geq q^{n-dk}$. $\square$

Proof (of Theorem 1). Let k such that $\text{rk}(F) = q^n - k$ ($k \geq 1$). If $k > \lfloor n/d \rfloor$ we are done. Otherwise, by Lemma 4 we have $k \geq q^{n-dk}$ hence, $\log_q(k) \geq n - dk$ and $(d+1)k \geq \log_q(k) + dk \geq n$ (because $k \geq \log_q(k)$). $\square$

Here is another application of Lemma 3.

Proposition 2. *Let $F \in \mathcal{F}(n,q,d)$ such that F is not constant. Then the number of preimages of any configuration is upper-bounded by $q^n - q^{n-d}$.*

Proof. Let $y \in Q^V$. Let us prove that $|F^{-1}(y)| \leq q^n - q^{n-d}$. Since F is not constant, there exists $z \in F(Q^V)$ such that $z_i \neq y_i$ for some $i \in V$. Since $F^{-1}([z_i]) \neq \emptyset$, by Lemma 3, $F^{-1}([z_i]) \geq q^{n-d}$. Furthermore, since $F^{-1}([z_i]) \cap F^{-1}(y) = \emptyset$, $|F^{-1}(y)| \leq q^n - q^{n-d}$. $\square$

It is tight because we can have $F(x) = 0^n$ if $x_{1,\dots,d} \neq 0^d$ and 10^{n-1} otherwise.

4 Realization results via feedback shift registers

In this section, we are interested in realizing examples of AN with *almost degree* 1, *i.e.*, whose all but one nodes have degree at most 1.

One important tool for this is the following. Let $g : \{0,1\}^n \to \{0,1\}$, and $F_g : \{0,1\}^n \to \{0,1\}^n$ be the corresponding *feedback shift register* (FSR), that is, $F_g(x) = F_g(x_1, \dots, x_n) = (x_2, \dots, x_{n-1}, g(x))$. $G(F_g)$ is thus obtained from the path $n \to n-1 \to \dots \to 1$ by adding an arc from i to 1 whenever g depends on input i: it has almost degree 1.

The *de Bruijn graph* of order n (over alphabet $\{0,1\}$) has set of vertices $V = \{0,1\}^n$ and set of arcs $E = \{(au, ub) : a, b \in \{0,1\}, u \in \{0,1\}^{n-1}\}$.

Proposition 3. *For any n and any $1 \leq k \leq 2^n$, the de Bruijn graph of order n admits a cycle of length k.*

Proof. The de Bruijn graph admits a *Eulerian* cycle because it is connected and all vertices have equal in- and out-degree. Since the Bruijn graph of order n+1 is the line digraph of the Bruijn graph of order n, we deduce that the de Bruijn graph admits a Hamiltonian cycle. Cycles of each length $0 < k < 2^n$ are a consequence of [5, Theorem 4]. □

Proposition 4. *For any n and any $0 \leq k \leq 2^n$, there exists $F : \{0,1\}^n \to \{0,1\}^n$ with almost degree 1 and whose maximum limit cycle has length k.*

Proof. Consider some cycle $C \subseteq \{0,1\}^n$ given by Proposition 3, and the feedback shift register F_g, where

$$g(x) = \begin{cases} b & \text{if } x = au \text{ and } au \to ub \in C \\ 0 & \text{otherwise.} \end{cases}$$

F_g has almost degree 1, and has the cycle C in its dynamics. To conclude the proof, it is sufficient to observe that the dynamics on the complement of C consists in adding 0 at node n and shifting node $i+1$ to node i for $i < n$. Therefore, the only possible cycle created by this part of the dynamics is possibly the fixed point $0 \cdots 0$. □

Now let us try to decrease the degree of the special vertex. Suppose that g is *additive*, *i.e.*, consider $\{0,1\}$ as the ring $\mathbb{Z}/2\mathbb{Z}$, and $g(x) = \sum_{i=1}^{n} a_i x_i$, for some coefficients $a_1, \ldots, a_n \in \{0,1\}$. In the corresponding interaction graph, $N^{\text{in}}(1) = \{i \mid a_i \neq 0\}$. The *characteristic polynomial* of g is $P = 1 + \sum_{i \in N^{\text{in}}(1)} X^i$. If P has degree n, then it is called *primitive* if it is irreducible and does not divide $X^k - 1$ for any $1 \leq k < 2^n - 1$. We say that P is a *trinomial* if it contains 3 terms, that is, if $P = X^n + X^k + 1$ for some $1 \leq k < n$.

Theorem 2 ([4]). *F_g has a limit cycle of length $2^n - 1$ and a fixed point if and only if P is* primitive *of degree n.*

Let us say that n is a *Mersenne exponent* if $2^n - 1$ is a (Mersenne) prime number.

Proposition 5. *For any Mersenne exponent $n \leq 3021377$, there is some order-n AN of degree 2 and almost degree* 1 *whose dynamics is the union of a limit cycle of length $2^n - 1$ and a fixed point.*

This corresponds to Example 1 and Figure 1(b).

Proof. If n is a Mersenne exponent and P has degree n, then P is primitive if and only if it is irreducible. For every Mersenne exponent $n \leq 3021377$, there exists at least one primitive trinomial P of degree n [1]. □

Example 1. Let $n = 5$, $q = 2$, and define the AN $F : \{0,1\}^5 \to \{0,1\}^5$ with $f_i(x) = x_{i-1}$ for $i \in \{2,3,4,5\}$, and $f_1(x) = x_3 \oplus x_5$ where $\oplus$ is the binary xor. Its dynamics has one fixed point and one cycle of length $2^n - 1$, while its interaction graph has degree 2 (see Figure 2), hence $F \in \mathcal{F}(5,2,2)$.

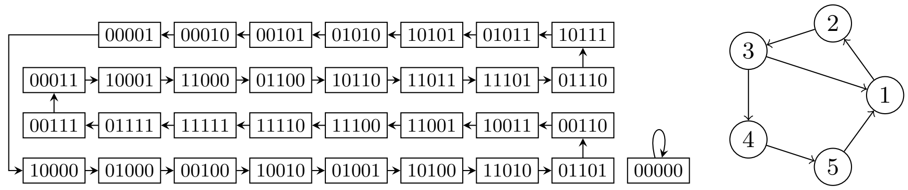


Fig. 2. Dynamics (left) and interaction graph (right) of the AN from Example 1.

5 Complexity of recognizing bounded-degree dynamics

Fix d and q, and consider the following decision problem called BDD (bounded-degree dynamics): given $F \in \mathcal{F}(n,q)$ represented by Boolean circuits, is there some $F' \in \mathcal{F}(n,q,d)$ such that $\mathcal{D}(F)$ and $\mathcal{D}(F')$ are isomorphic?

Theorem 3. *The problem BDD is in PSPACE for every d, q, and co-NP-hard for any $q \geq 2$ and $d \geq 1$.*

Proof. For the upper bound, a naive algorithm solving BDD consists in guessing $F' \in \mathcal{F}(n,q,d)$ (whose size is polynomial in F thanks to the bounded-degree condition) and checking that $\mathcal{D}(F)$ and $\mathcal{D}(F')$ are isomorphic. Given that planar graph isomorphism is computable with a LOGSPACE Turing machine M [3] and that $\mathcal{D}(F)$ and $\mathcal{D}(F')$ are at most exponentially larger than the input (Boolean circuit for F), we can test isomorphism of $\mathcal{D}(F)$ and $\mathcal{D}(F')$ in PSPACE by simulating each reading step of the read-only input tape of M by an evaluation of circuit in polynomial time (testing $F(x) = y$ is the same as testing the presence of the corresponding arc in $\mathcal{D}(F)$). This gives an algorithm in NP with an oracle in PSPACE, *i.e.*, an algorithm in the complexity class PSPACE.

For the co-NP-hardness we reduce from UNSAT. Given a propositional formula ϕ on p variables $v_1, \dots, v_p$, we construct $F \in \mathcal{F}(n,q)$ on $|V| = p + d$ automata, with $P = \{v_1, \dots, v_p\}$, $D = \{t_1, \dots, t_d\}$ and $V = P \cup D$. Let $Q = \{0, \dots, q-1\}$, and for $x \in Q^V$, consider the valuation $\theta(x_P)$ sending each 0 to false and other symbols to true. Set the local functions to be the identity $f_i(x) = x_i$ for every $i \in V \setminus \{t_d\}$, and:

$$f_{t_d}(x) = \begin{cases} x_{t_d} + 1 \mod q & \text{if } x_D = a^d \text{ and } \phi(\theta(x_P)), \\ x_{t_d} & \text{otherwise.} \end{cases}$$

If ϕ is unsatisfiable, then t_d depends only on D and F has degree d, hence it is a positive instance of BDD. Otherwise, F is not the identity, and it has:

- $(q^d - 1)q^p = q^n - q^{n-d}$ fixed points with $x_D \neq a^d$,
- at least one additional fixed point with $x_D = a^d$ and $\theta(x_P)$ satisfying ϕ.

Proposition 1 then implies that it is a negative instance of BDD. □

If we drop the isomorphism condition from the above problem, we get another one called BDIG (bounded-degree interaction graph): given $F \in \mathcal{F}(n,q)$ represented by Boolean circuits, is there some $F' \in \mathcal{F}(n,q,d)$ such that $\mathcal{D}(F) = \mathcal{D}(F')$? or, equivalently, is the degree of the interaction graph of F bounded by d?

Theorem 4. *The problem BDIG is co-NP-complete.*

Proof. The lower bound is given by the same reduction as in the proof of Theorem 3. For the upper bound, a simple co-NP algorithm consists in guessing an automaton $i \in V$, $d+1$ configurations $x^1, \ldots, x^{d+1}$, and $d+1$ distinct automata $i_1, \ldots, i_{d+1}$, then checking for each $j \in \{1, \ldots, d+1\}$ that $f_i(x^j) \neq f_i(x^j + e_{i_j})$. For each j, it checks whether x^j witnesses the effective dependency of i on automaton i_j. It is possible to guess $d+1$ such witnesses if and only if the interaction graph of F has degree at least $d+1$. □

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