An extension of a combinatorial fixed point theorem of Shih and Dong

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Dedicated to Professor Bruno Soubeyran with admiration and affection

Abstract: Shih and Dong have proved a boolean analogue of the Jacobian problem: if a map from $\{0,1\}^n$ to itself has the property that all the boolean eigenvalues of the discrete Jacobian matrix of each element of $\{0,1\}^n$ are zero, then it has a unique fixed point. In this note, this result is extended to any map F from the product X of n finite intervals of integers to itself. Our method of proof reveals an interesting property of the asynchronous state graph of F used to model the qualitative behavior of genetic regulatory networks.

Key words: Discrete dynamical systems, Asynchronous automata networks, Jacobian conjecture, Discrete Jacobian Matrix, Boolean eigenvalue, Fixed point, Genetic regulatory networks.

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1 Introduction

In the course of his analysis of discrete iterations, Robert introduced a discrete Jacobian matrix for boolean maps and the notion of boolean eigenvalue [2, 3, 4, 5]. This material allows Shih and Ho to state in 1999 a boolean analogue of the Jacobian conjecture [7]: if a map from $\{0,1\}^n$ to itself is such that all the boolean eigenvalues of the discrete Jacobian matrix of each element of $\{0,1\}^n$ are zero, then it has a unique fixed point. Thanks to the work of Shih and Dong [6], this conjecture is now a theorem.

In order to extend this theorem from the boolean case to the finite discrete case, we introduce here a discrete Jacobian matrix for maps from the product X of n finite intervals of integers to itself which generalizes the Robert's one. More precisely, given a map F from X to itself, we define a discrete Jacobian matrix $F'(x,v)$ depending on an element $x \in X$ and a variation vector $v \in \{-1,1\}^n$ such that $x+v \in X$. Then, we state and prove the following extension of the Shih-Dong's fixed point theorem: if F has the property that for each $x \in X$ and $v \in \{-1,1\}^n$ such that $x+v \in X$, all the boolean eigenvalues of $F'(x,v)$ are zero, then F has a unique fixed point.

Our method of proof reveals that this sufficient condition for F to have a unique fixed point is also sufficient for the presence of a shortest path from any element of X to the fixed point of F in the asynchronous state graph of F. This asynchronous state graph is often used to model the behavior of genetic regulatory networks (see the work of Thomas; [8, 9] for instance), and our motivation to extend the Shih-Dong's fixed point theorem to the discrete case comes from the fact that, in this biological context, X is generally not reduced to the n-dimensional hypercube $\{0,1\}^n$.

2 Discrete Jacobian matrix and boolean spectral radius

In this section, we state some definitions needed to formulate the main result.

2.1 Discrete Jacobian matrix

Let n be a positive integer and let $X = X_1 \times \ldots \times X_n$ be the product of n finite intervals of integers of cardinality ≥ 2. We denoted by $V(X)$ the set of couples (x, v) such that $x \in X$, $v \in \{-1, 1\}^n$ and $x + v \in X$. Furthermore, for all $i \in \{1, \ldots, n\}$, we denote by e_i the n-tuple whose ith component is 1 and whose other components are 0.

We are now in position to introduce our notion of discrete Jacobian matrix. Given a map $F = (f_1, \ldots, f_n)$ from X to itself and $(x, v) \in V(X)$, we call *discrete Jacobian matrix* of F evaluated at x with the variation vector v, and we denote by $F'(x, v) = (f_{ij}(x, v))$, the $n \times n$ matrix over $\{0, 1\}$ defined by:

$$f_{ij}(x, v) = \begin{cases} 1 & \text{if } f_i(x) \text{ and } f_i(x + v_j e_j) \text{ are on both sides of } x_i + v_i/2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(Integers a and b are *on both sides* of a non-integer number c if $a < c < b$ or $b < c < a$.)

Note that this discrete Jacobian matrix has been introduced by Robert [3, 4, 5] in the boolean case, *i.e.* when $X = \{0, 1\}^n$. (The Robert's discrete Jacobian matrix is usually simply denoted $F'(x)$ since for each $x \in \{0, 1\}^n$ there is a unique $v \in \{-1, 1\}^n$ such that $x + v \in \{0, 1\}^n$.) Note also that in the boolean case, the condition "$f_i(x)$ and $f_i(x + v_j e_j)$ are on both sides of $x_i + v_i/2$", which is discussed below (cf. Remark 1), is equivalent to the condition $f_i(x) \neq f_i(x + v_j e_j)$.

2.2 Boolean spectral radius

Let $B = (b_{ij})$ be an $n \times n$ matrix over $\{0, 1\}$. The *boolean eigenvalues* of B have been defined by Robert [2, 4, 5], and the *boolean spectral radius* of B, denoted $\rho(B)$, is then defined as the largest boolean eigenvalue of B [2, 4, 5]. Following [6], $\rho(B)$ can be characterized with elementary graph-theoretic notions. Let $\Gamma(B)$ denotes the directed graph whose set of nodes is $\{1, \ldots, n\}$ and such that there is an edge from j to i if $b_{ij} = 1$, *i.e* the directed graph whose adjacency matrix is the transposite of B. Then, $\rho(B) = 1$ if $\Gamma(B)$ has an oriented cycle and $\rho(B) = 0$ otherwise.

2.3 Main result

The aim of this note is to prove the following theorem.

Theorem 1 (Combinatorial Fixed Point Theorem)

Let X be the product of n finite intervals of integers of cardinality ≥ 2. If a map F from X to itself is such that $\rho(F'(x, v)) = 0$ for all $(x, v) \in V(X)$, then it has a unique fixed point.

For $X = \{0, 1\}^n$, this theorem is equivalent to the boolean analogue of the Jacobian conjecture stated by Shih and Ho [7] and proved by Shih and Dong [6]. Theorem 1 can thus be viewed as a discrete version of the Jacobian conjecture.

Remark 1 In the definition of the discrete Jacobian matrix, the condition "$f_i(x)$ and $f_i(x+v_je_j)$ are on both sides of $x_i+v_i/2$" may seem not natural. A more natural condition leads to the *usual discrete Jacobian matrix* $F'[x, v] = (f_{ij}[x, v])$ defined by:

$$f_{ij}[x, v] = \begin{cases} 1 \text{ if } f_i(x) \neq f_i(x + v_je_j), \\ 0 \text{ otherwise.} \end{cases}$$

However, if $f_{ij}(x, v) = 1$ then $f_{ij}[x, v] = 1$ and thus, $\Gamma(F'(x, v))$ is a subgraph of $\Gamma(F'[x, v])$. Therefore, if $\Gamma(F'[x, v])$ has no oriented cycle then $\Gamma(F'(x, v))$ has no oriented cycle and it follows that the condition "$\rho(F'[x, v]) = 0$ for all $(x, v) \in V(X)$" implies the condition of Theorem 1. So, *if we use the usual discrete Jacobian matrix instead of the discrete Jacobian matrix defined in (1), then Theorem 1 remains valid but becomes less strong*. For instance, if $n = 1$ and $X = \{0, 1, \ldots, 9\}$, then there are only 10 maps F from X to itself such that $\rho(F'[x, v]) = 0$ for all $(x, v) \in V(X)$ (they correspond to the 10 constant maps from X to itself which have obviously a unique fixed point) whereas there are more than 186 millions of maps F such that $\rho(F'(x, v)) = 0$ for all $(x, v) \in V(X)$. Another example showing that Theorem 1 is less strong if we use the usual discrete Jacobian matrix is given at the end of this note.

3 Asynchronous state graph and path lemma

Let X be the product of n finite intervals of integers of cardinality ≥ 2, and let F be a map from X to itself. In this section, we prove the key lemma which is both used to prove, under the conditions of Theorem 1, the existence and the uniqueness of a fixed point for F.

To state and prove this lemma, we use a supplementary definition: the *asynchronous state graph* of F, denoted $A(F)$, is the directed graph whose set of nodes is X and such that there is an edge from x to y if there exists $i \in \{1, \ldots, n\}$ such that:

$$f_i(x) \neq x_i \qquad \text{and} \qquad y = x + sign(f_i(x) - x_i) \cdot e_i,$$

where $sign(a) = a/|a|$ for all integers $a \neq 0$. Note that $x \in X$ is a fixed point of F if and only if x has no successor in $A(F)$. In the sequel, by convention, we suppose that there is a path of length zero from each node to itself in $A(F)$.

Lemma 1 (Path lemma)

Let F be a map from X to itself such that $\rho(F'(x,v)) = 0$ for all $(x,v) \in V(X)$, and let $x, y \in X$. If $f_i(x) \leq x_i \leq y_i$ or $y_i \leq x_i \leq f_i(x)$ for all $i \in \{1,\dots,n\}$, then there is a path from y to x in the asynchronous state graph of F.

Proof − We reason by induction on the Manhattan distance

$$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

between x and y. The base case is trivial: if $d(x,y) = 0$ then $x = y$ thus there is a path (of length zero) from y to x. For the induction step, we suppose that $d(x,y) > 0$ and that the lemma holds for each couple $(x', y') \in X \times X$ such that $d(x', y') < d(x,y)$. Let I be the set of indices $i \in \{1,\dots,n\}$ such that $x_i \neq y_i$, and let $v \in \{-1,1\}^n$ be such that $(x,v) \in V(X)$ and:

$$\forall i \in I, \qquad v_i = sign(y_i - x_i). \tag{2}$$

(Such a v necessarily exists.) Then, $(x,v) \in V(X)$. Moreover, there exists $j \in I$ such that $f_{ij}(x,v) = 0$ for all $i \in I$. Indeed, if for all $j \in I$ there exists $i \in I$ such that $f_{ij}(x,v) = 1$, then $\Gamma(F'(x,v))$ has an oriented cycle involving only nodes of I, thus $\rho(F'(x,v)) = 1$, a contradiction. So let $j \in I$ be such that $f_{ij}(x,v) = 0$ for all $i \in I$, and let $z = x + v_j e_j$. Then, there is an edge from z to x in the asynchronous state graph of F. Indeed, following the conditions of the lemma and (2), we have:

$$\begin{cases} f_j(x) \leq x_j < z_j \leq y_j & \text{if } v_j = +1, \\ y_j \leq z_j < x_j \leq f_j(x) & \text{if } v_j = -1. \end{cases}$$

Since $f_{jj}(x, v) = 0$, we deduce that:

$$\begin{cases} f_j(z) \leq x_j < z_j \leq y_j & \text{if } v_j = +1, \\ y_j \leq z_j < x_j \leq f_j(z) & \text{if } v_j = -1. \end{cases} \tag{3}$$

Consequently, $sign(f_j(z) - z_j) = -v_j$ and since $x = z - v_j e_j$ there is an edge from z to x in $A(F)$. To complete the proof, it is thus sufficient to show that there is a path from y to z in $A(F)$. Following the conditions of the lemma and (2), we have:

$$\forall i \in I \setminus \{j\}, \qquad \begin{cases} f_i(x) \leq x_i = z_i < y_i & \text{if } v_i = +1, \\ y_i < z_i = x_i \leq f_i(x) & \text{if } v_i = -1. \end{cases}$$

Since $f_{ij}(x, v) = 0$ for all $i \in I$, we deduce that:

$$\forall i \in I \setminus \{j\}, \qquad \begin{cases} f_i(z) \leq x_i = z_i < y_i & \text{if } v_i = +1, \\ y_i < z_i = x_i \leq f_i(z) & \text{if } v_i = -1. \end{cases}$$

From this and (3), we deduce that, for all $i \in \{1, \ldots, n\}$, $f_i(z) \leq z_i \leq y_i$ or $y_i \leq z_i \leq f_i(z)$. Moreover, we deduce from (2) that $d(z, y) = d(x, y) - 1$. Thus, by induction hypothesis, there is a path from y to z in $A(F)$. $\square$

Remark 2

1. From this proof, it is easy to show that, under the conditions of the lemma, there is a

path from y to x of length $d(x, y)$. Indeed, this holds trivially if $x = y$, and otherwise, if we suppose that the path from y to z resulting from the induction hypothesis is of length $d(z, y)$ then, since $d(z, y) = d(x, y) - 1$ and since there is an edge from z to x, there is a path from y to x of length $d(x, y)$. Note that such a path is a *shortest path* in the sense that any path from y to x in $A(F)$ is of length $\geq d(x, y)$.

2. Richard and Comet [1] proved the presence of a path from y to x of length $d(x, y)$ under quite less strong conditions with additional technical arguments. We give here the short proof of the previous lemma in order to present all the arguments needed to extend the theorem of Shih and Dong [6] to Theorem 1, and to make this note as self-contained as possible.

3. The asynchronous state graph of F is often used to model the qualitative behavior of gene regulatory networks (see the work of Thomas; [8, 9] for instance). In this context, $X = X_1 \times \ldots \times X_n$ corresponds to the set of possible states of the network, each interval X_i corresponds to the set of possible expression levels of the gene i, and each path of $A(F)$ corresponds to a possible evolution of the network. Generally, if the protein encoded by a gene i regulates the expression of $m > 0$ genes, then the interval X_i used to model the activity of gene i is $\{0, 1, \ldots, m\}$ [8, 9]. Since it is very frequent that $m > 1$, X is generally not reduced to the n-dimensional hypercube $\{0, 1\}^n$.

4. It is easy to show that the Jacobian matrix of F only depends on its asynchronous state graph, *i.e.* that if $A(F) = A(G)$ then $F'(x, v) = G'(x, v)$ for all $(x, v) \in V(X)$.

4 Proof of Theorem 1

Let $\mathbb{X}$ be the set of all the products of n finite intervals of integers of cardinality ≥ 2. We said that an element X of $\mathbb{X}$ *has the property* (J) if Theorem 1 holds for all maps F from X to itself. So Theorem 1 is true if and only if any element of $\mathbb{X}$ has the property (J). We prove this by induction on the set $\mathbb{X}$ ordered by the inclusion relation $\subset$.

4.1 Base case

An element $X = X_1 \times \cdots \times X_n$ of $\mathbb{X}$ is minimal with respect to $\subset$ if and only if $|X_i| = 2$ for all $i \in \{1, \ldots, n\}$. So, any minimal element of $(\mathbb{X}, \subset)$ can be identified to $\{0,1\}^n$ and, following the Shih-Dong's fixed point theorem [6], has the property (J).

4.2 Induction step

Let $X = X_1 \times \cdots \times X_n$ be a non minimal element of $(\mathbb{X}, \subset)$, and suppose that any $Y \in \mathbb{X}$ strictly included in X has the property (J). Let $F = (f_1, \ldots, f_n)$ be a map from X to itself such that $\rho(F'(x, v)) = 0$ for all $(x, v) \in V(X)$. We want to prove that F has a unique fixed point.

We first prove, by contradiction, that F has at least one fixed point. We thus suppose that F has no fixed point. Since X is not a minimal element of $(\mathbb{X}, \subset)$, there exists $i \in \{1, \ldots, n\}$ such that $|X_i| > 2$. Without loss of generality, suppose that $|X_1| > 2$. Let

$$a = \min(X_1) \quad \text{and} \quad b = \max(X_1).$$

In order to use the induction hypothesis, consider the map $\tilde{F} = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_n)$ from $\tilde{X} =$

$X_1 \setminus \{a\} \times X_2 \times \cdots \times X_n$ to itself defined by:

$$\forall x \in \tilde{X}, \quad \tilde{F}(x) = (\max(f_1(x), a+1), f_2(x), \ldots, f_n(x)).$$

Let $x, y \in \tilde{X}$. It is clear that $f_1(x) \leq \tilde{f}_1(x)$. Moreover, if $\tilde{f}_1(x) < \tilde{f}_1(y)$ then $a+1 < \tilde{f}_1(y)$ thus $\tilde{f}_1(y) = f_1(y)$. Because, for all $i \in \{2, \ldots, n\}$, $\tilde{f}_i(x) = f_i(x)$ and $\tilde{f}_i(y) = f_i(y)$, we deduce that, for all $i \in \{1, \ldots, n\}$:

$$\tilde{f}_i(x) < \tilde{f}_i(y) \Rightarrow f_i(x) \leq \tilde{f}_i(x) < \tilde{f}_i(y) \leq f_i(y). \tag{4}$$

It follows that, for all $(x, v) \in V(\tilde{X})$ and $i, j \in \{1, \ldots, n\}$:

$$\tilde{f}_{ij}(x, v) = 1 \Rightarrow f_{ij}(x, v) = 1.$$

Thus, $\Gamma(\tilde{F}'(x, v))$ is a subgraph of $\Gamma(F'(x, v))$. By hypothesis, $\rho(F'(x, v)) = 0$ thus $\Gamma(F'(x, v))$ has no oriented cycle and we deduce that its subgraph $\Gamma(\tilde{F}'(x, v))$ has no oriented cycle. Thus $\rho(\tilde{F}'(x, v)) = 0$, and this holds for all $(x, v) \in V(\tilde{X})$. Since $\tilde{X}$ is strictly included in X, by induction hypothesis, $\tilde{X}$ has the property (J), and we deduce that $\tilde{F}$ has a (unique) fixed point. So let $\tilde{x}$ be a fixed point of $\tilde{F}$. Clearly:

$$\forall i \in \{2, \ldots, n\}, \quad \tilde{x}_i = \tilde{f}_i(\tilde{x}) = f_i(\tilde{x}). \tag{5}$$

Because $\tilde{x}$ is not a fixed point of F, we deduce that $\tilde{x}_1 = \tilde{f}_1(\tilde{x}) \neq f_1(\tilde{x})$. Thus:

$$f_1(\tilde{x}) = a < \tilde{f}_1(\tilde{x}) = a + 1 = \tilde{x}_1 < b. \tag{6}$$

Now, consider the map $\bar{F} = (\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n)$ from $\bar{X} = X_1 \setminus \{b\} \times X_2 \times \cdots \times X_n$ to itself defined by:

$$\forall x \in \bar{X}, \quad \bar{F}(x) = (\min(f_1(x), b-1), f_2(x), \ldots, f_n(x)).$$

With similar arguments, we prove that $\rho(\bar{F}(x,v)) = 0$ for all $(x,v) \in V(\bar{X})$ and that $\bar{F}$ has a (unique) fixed point $\bar{x}$ such that:

$$\bar{x}_1 = b - 1 < f_1(\bar{x}) = b. \tag{7}$$

According to (6), we have $\tilde{x} \in \bar{X}$. From (5), we deduce that:

$$\forall i \in \{2, \ldots, n\}, \quad \bar{f}_i(\tilde{x}) = f_i(\tilde{x}) = \tilde{x}_i, \tag{8}$$

and from (6) and (7) we deduce that:

$$f_1(\tilde{x}) = \bar{f}_1(\tilde{x}) < \tilde{x}_1 \le \bar{x}_1.$$

From this and (8), it is clear that:

$$\forall i \in \{1, \ldots, n\}, \quad \bar{f}_i(\tilde{x}) \le \tilde{x}_i \le \bar{x}_i \quad \text{or} \quad \bar{x}_i \le \tilde{x}_i \le \bar{f}_i(\tilde{x}).$$

Therefore, following the path lemma, there is a path from $\bar{x}$ to $\tilde{x}$ in the asynchronous state graph of $\bar{F}$. Following (6) and (7), we have $f_1(\tilde{x}) \neq f_1(\bar{x})$ thus $\tilde{x} \neq \bar{x}$. So the existence of a path from $\bar{x}$ to $\tilde{x}$ implies that $\bar{x}$ has a successor in $A(\bar{F})$. Therefore, $\bar{x}$ is not a fixed point of $\bar{F}$, a contradiction. It means that F has at least one fixed point.

So let x be a fixed point of F and let y be an element of X different than x. According to the path lemma, there is a path from y to x in the asynchronous state graph of F so y is not a fixed point of F. Thus x is the unique fixed point of F and consequently, X has the property (J). This completes the proof of Theorem 1.

5 Concluding remarks

The spectral condition "$\rho(F'(x,v)) = 0$ for all $(x,v) \in V(X)$" implies that F has a unique fixed point, and it also implies the presence of a shortest path from any element of X to this fixed point in the asynchronous state graph of F used to model the behavior of genetic regulatory networks. So, under the spectral condition, there is, in the asynchronous state graph of F, a kind of convergence toward a unique fixed point, but this convergence is weak in the sense that any path does not lead necessarily to the fixed point, as shown by the following example.

Example $n = 2$ and $X = \{0,1,2\} \times \{0,1,2\}$. The map F and its asynchronous state graph are the following:

x	$F(x)$
$(0,0)$	$(0,2)$
$(0,1)$	$(2,2)$
$(0,2)$	$(2,2)$
$(1,0)$	$(0,2)$
$(1,1)$	$(1,1)$
$(1,2)$	$(2,0)$
$(2,0)$	$(0,0)$
$(2,1)$	$(0,0)$
$(2,2)$	$(2,0)$

$A(F)$

$$\begin{tikzcd}
(0,2) \arrow[r] & (1,2) \arrow[r] \arrow[d] & (2,2) \arrow[d] \\
(0,1) \arrow[u] \arrow[r] & (1,1) & (2,1) \arrow[l] \arrow[d] \\
(0,0) \arrow[u] & (1,0) \arrow[l] \arrow[u] & (2,0) \arrow[l]
\end{tikzcd}$$

It is straightforward to show that, for all $(x,v) \in V(X)$:

$$F'(x,v) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad F'(x,v) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Thus, for all $(x,v) \in V(X)$, $\rho(F'(x,v)) = 0$. We can see that F has actually a unique fixed point and that there is a shortest path from any point of X to this fixed point in $A(F)$. However, $A(F)$ has an oriented cycle so there are paths which do not lead to the fixed point of F.

Note also that, for this example, the condition "$\rho(F'[x,v]) = 0$ for all $(x,v) \in V(X)$", where $F'[x,v]$ is the usual discrete Jacobian matrix introduced in the first remark, does not hold. Indeed, we have, whenever $x = (1, 1)$ and $v \in \{-1, 1\}^2$:

$$F'[x, v] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and thus $\rho(F'[x, v]) = 1$ (since $\Gamma(F'[x, v])$ contains three oriented cycles).

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