Asynchronous simulation of Boolean networks by monotone Boolean networks

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A **Boolean network** with n components is a function

$$\begin{aligned} f : \{0,1\}^n &\to \{0,1\}^n \\ x = (x_1, \ldots, x_n) &\mapsto f(x) = (f_1(x), \ldots, f_n(x)) \end{aligned}$$

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Interaction graph: digraph on $\{1, \ldots, n\}$ such that

$$j \to i \iff f_i \text{ depends on } x_j$$

Example

3-component net f

$$f_1(x) = \overline{x_1} + x_3$$
$$f_2(x) = x_1 + x_3$$
$$f_3(x) = x_1 x_2 x_3$$

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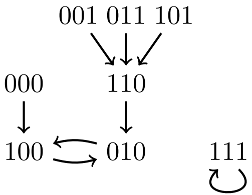

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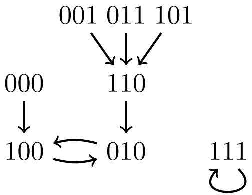

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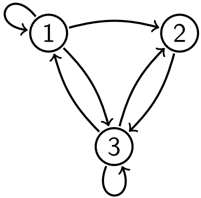

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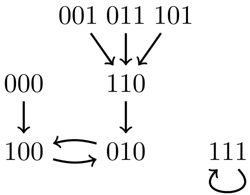

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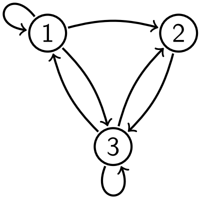

Many applications

- Neural networks [McCulloch & Pitts 1943]
- **Gene networks** [Kauffman 1969, Thomas 1973]

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Given a configuration x^0 and an infinite sequence $i_0, i_1, i_2, \ldots$ of components to update, the resulting **asynchronous trajectory** is

$$x^{t+1} = (x_1^t, \ldots, \underset{\substack{\uparrow \\ \text{update of } i_t \text{ only}}}{f_{i_t}(x^t)}, \ldots, x_n^t)$$

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Asynchronous graph: digraph on $\{0,1\}^n$ such that, for all x and i,

$$x \ \rightarrow \ (x_1, \ldots, f_i(x), \ldots, x_n)$$

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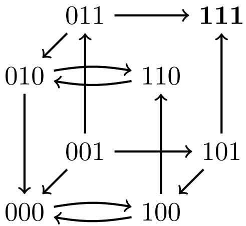
Asynchronous graph $\Gamma(f)$



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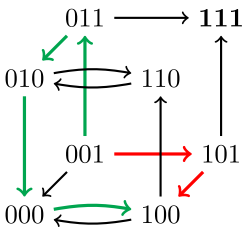

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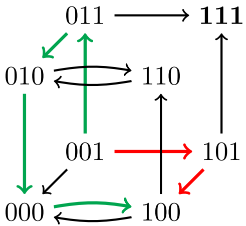

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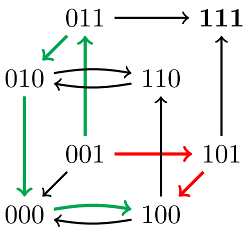

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Remark: Every geodesic is a shortest path of length at most n.

A network $f : \{0,1\}^n \to \{0,1\}^n$ is **monotone** if, for all $x, y \in \{0,1\}^n$,

$$x \leq y \quad \Rightarrow \quad f(x) \leq f(y)$$

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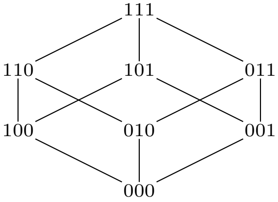
$\{0,1\}^n$ with the componentwise natural order $\leq$ is the **Boolean lattice**.



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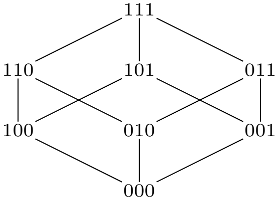

Monotone networks are interesting, both from the theoretical and practical point of view. Many results on **fixed points.**

Theorem [Knaster-Tarski 1928]

If f is monotone, then the set of fixed points of f is a non-empty lattice.

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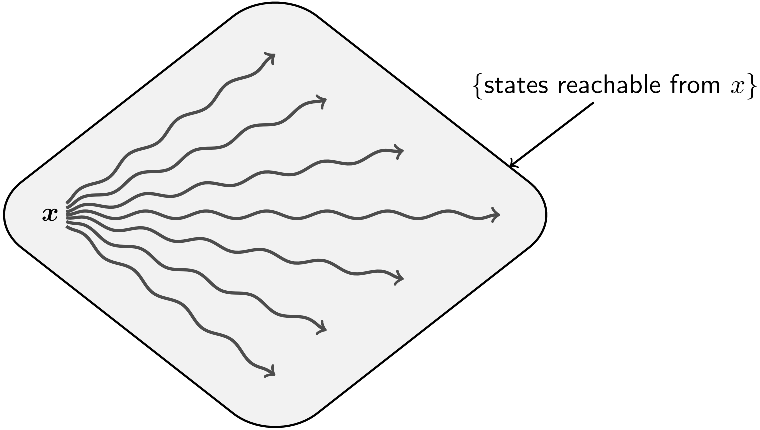
Remark: the fixed point z and the geodesic can be computed in $O(n^2)$.

Asynchronous reachability in monotone boolean networks



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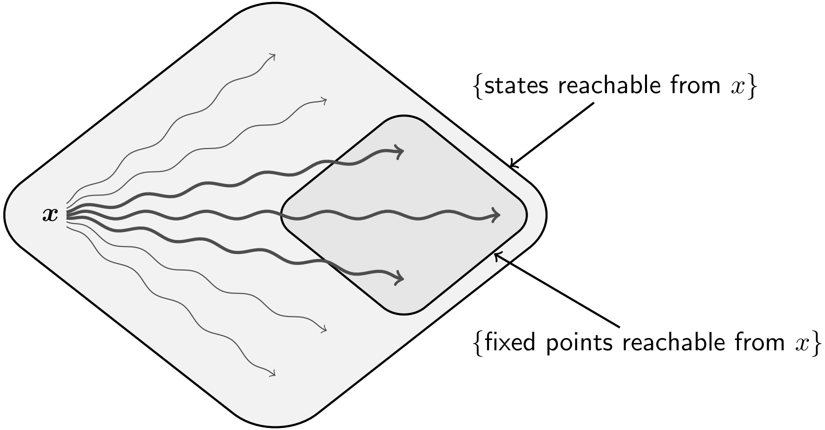

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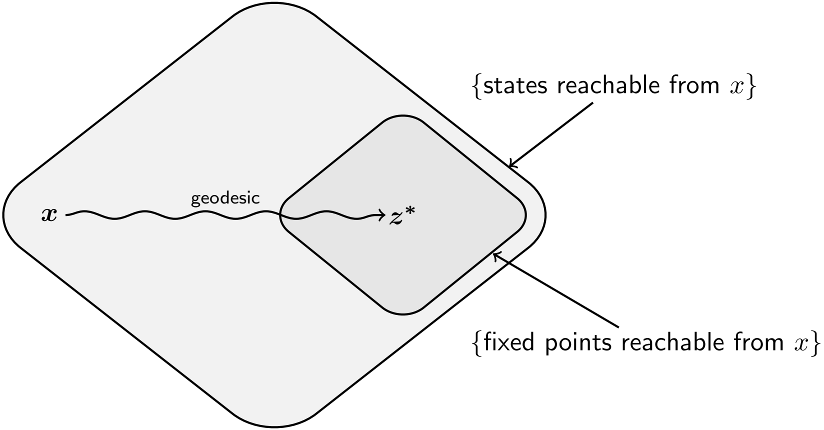

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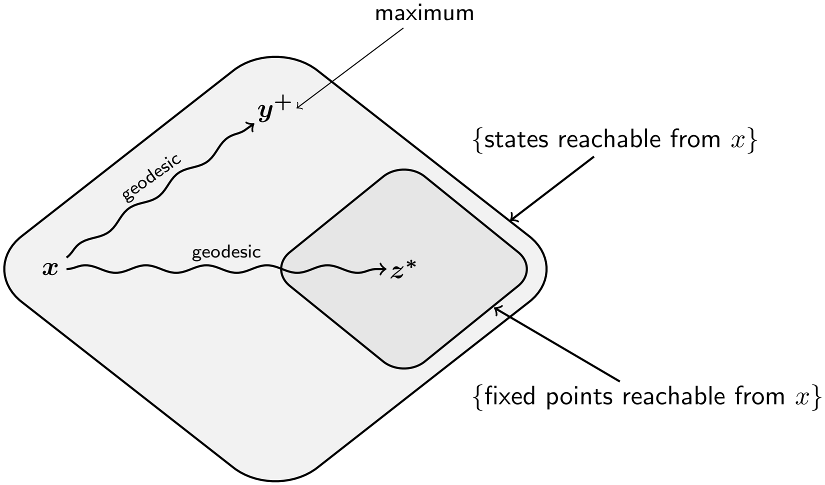

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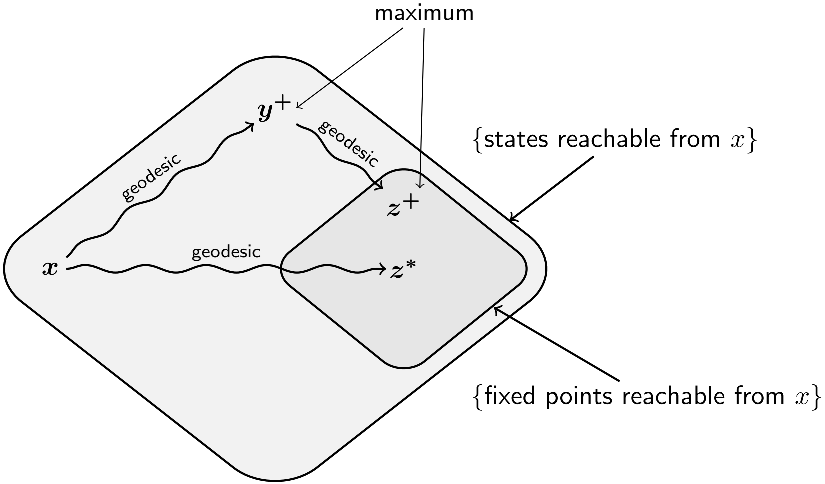

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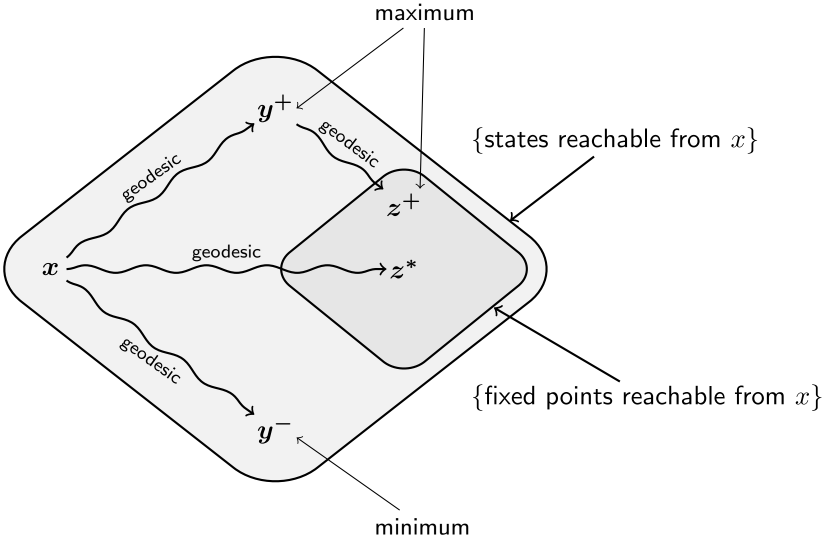

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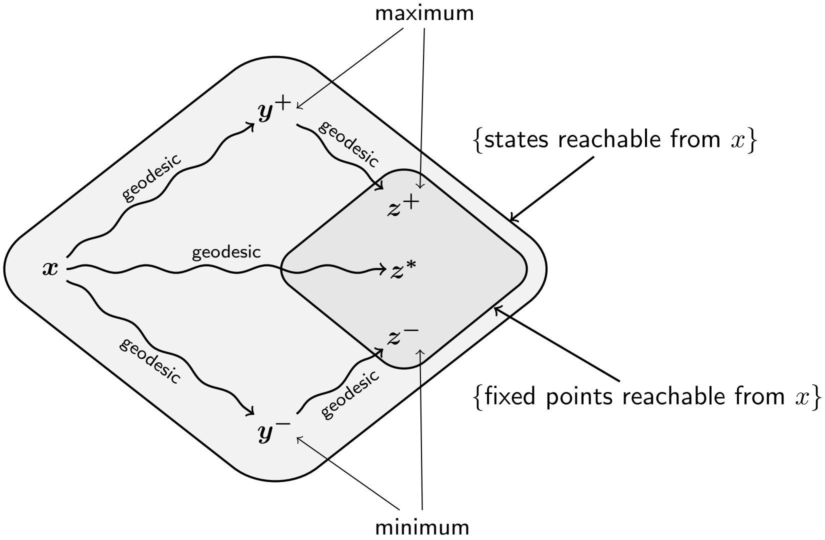
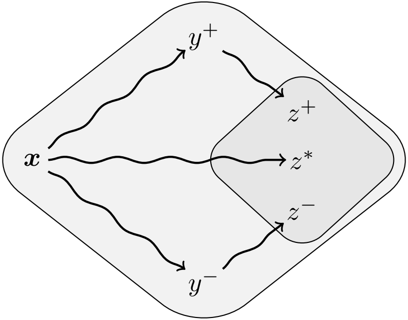
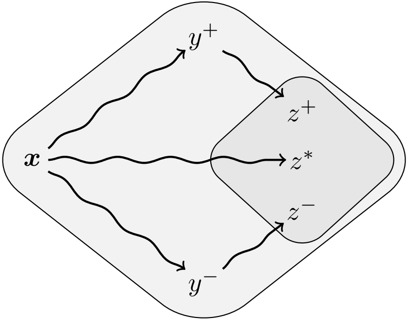



y^+
x
z^+
z^*
z^-
y^-



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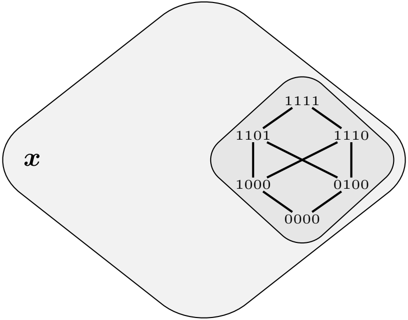

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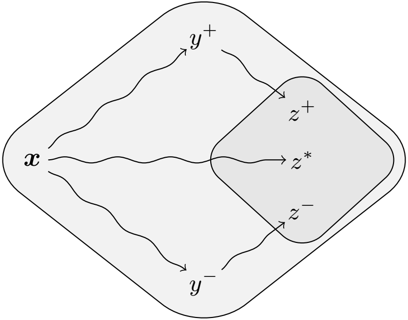
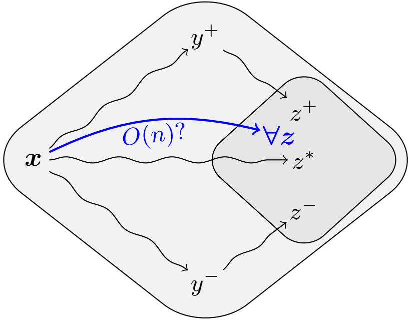


Question 2. Is there, for every reachable fixed point z, a "short path" from x to z?



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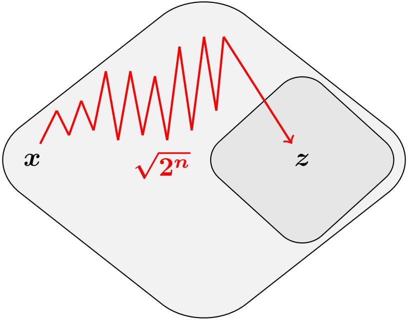

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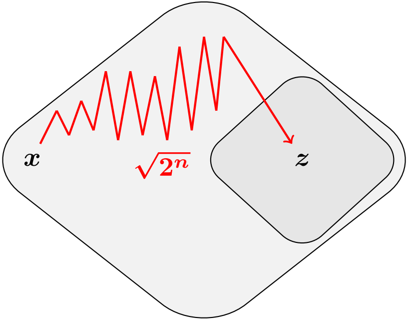

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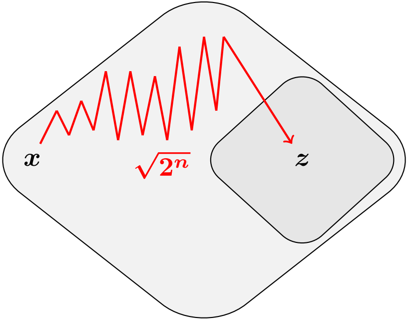
Theorem 1

For every n, there is a n-component monotone network f with a shortest path of length at least $\sqrt{2^n}$ that ends with a fixed point.



$\mathbf{diam}(\Gamma(f)) \geq \sqrt{2^n}$

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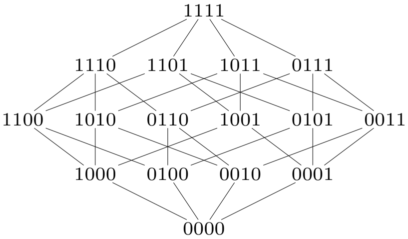
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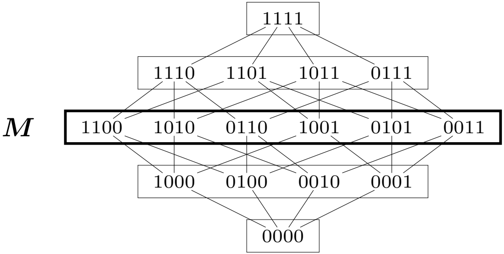
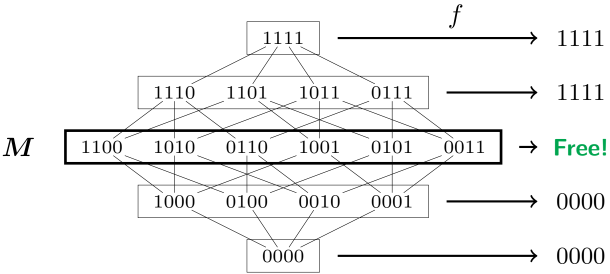


$|M| \geq 2^{n - \log n - 1}$



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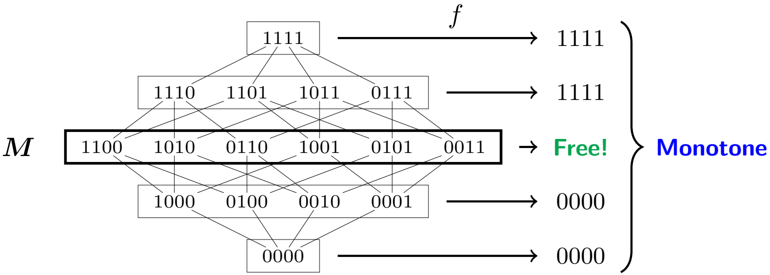

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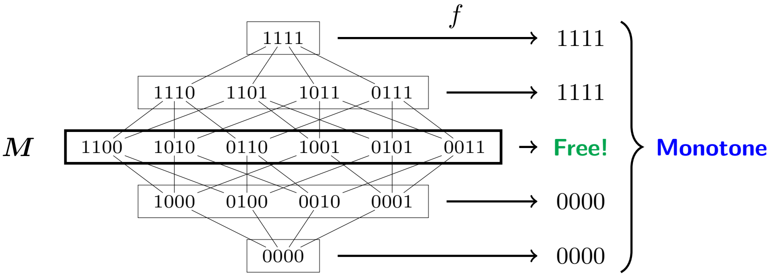
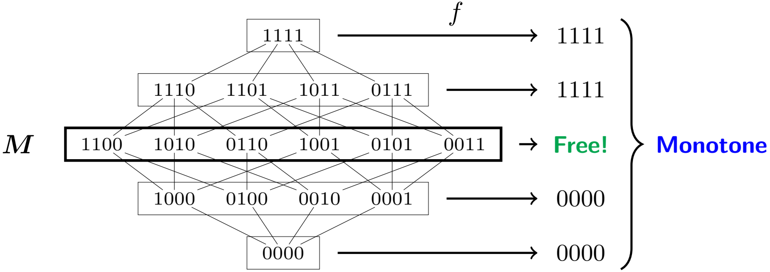


Since $|M| \geq 2^{n-\log n-1}$ we deduce that

An n-component monotone network f may simulate the synchronous dynamics of every $\lfloor n - \log n - 1 \rfloor$-component network h.



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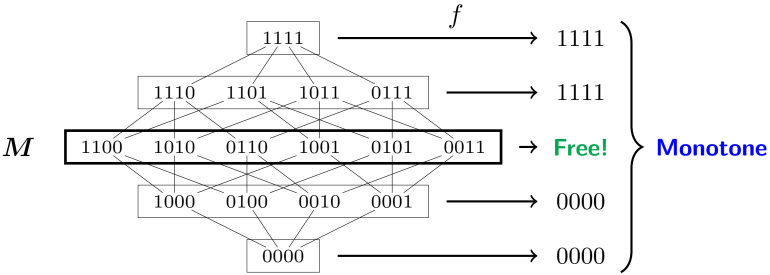

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If $\Gamma(h)$ is an Hamiltonian path (length $= 2^{\frac{n}{2}} - 1$) then f has an shortest path of length at least $2^{\frac{n}{2}}$ that ends with a fixed point (Theorem 1).

Open problems

1. Let diam(n) be the maximal diameter of the asynchronous graph of a monotone network with n components.

$$\sqrt{2^n} \leq \mathrm{diam}(n) \leq 2^n - 2$$

   Is there a close formula for diam(n)? Or at least better bounds?

2. Is it more easy to enumerate all fixed points reachable from x when the diameter is small?

3. Does a large diameter force some structures in the interaction graph? Such as long cycle or many disjoint cycles?


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
شكرا
obrigado
dziękuję
bedankt
спасибо
Баярлалаа
рахмат
sukriya
kop khun krap
terima kasih
go raibh maith agat
mochchakkeram
arigatô
dakujem
мерси
grazie
dank je
tapadh leat
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ありがとう
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vinaka
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спасибі
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dankie
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dhanyavad
bayarlalaa
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