Dividing permutations in the semiring of functional digraphs

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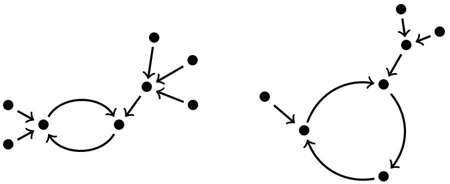
AUTOMATA'24

Durham, UK, July 22

Functional digraphs

Each vertex has exactly one out-neighbor



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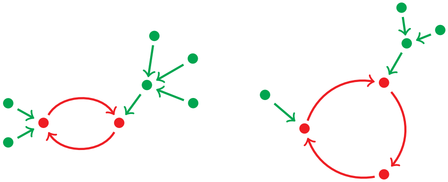

- Periodic part = disjoint union of cycles = **permutation**
- Transient part

Semiring on functional digraphs [Dennunzio, Dorigatti, Formenti, Manzoni, Porreca 2018]

Given two functional digraphs A, B:

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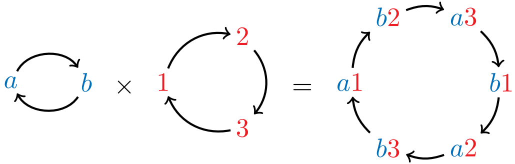

$$C_2 \times C_3 = C_6$$

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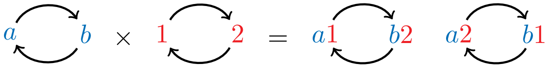

$$C_2 \times C_2 = C_2 + C_2 = 2C_2$$

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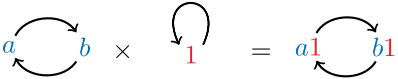

$$C_2 \times C_1 = C_2$$

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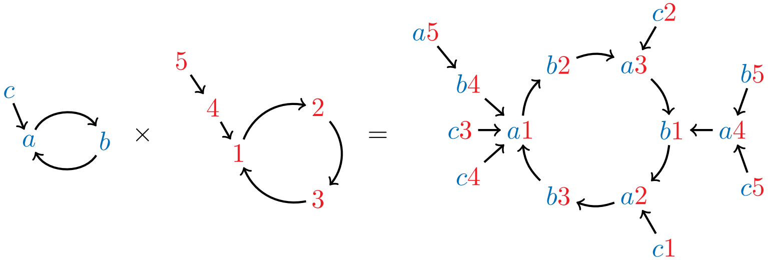

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$$\underbrace{C_2}_{A} \times \underbrace{C_2}_{X} = \underbrace{2C_2}_{B}$$
$$\underbrace{C_2}_{A} \times \underbrace{2C_1}_{X} = \underbrace{2C_2}_{B}$$

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Polynomial algorithm to decide if $A \mid B$ when

- B is a dendron. [Naquin, Gadouleau 24]
- A, B are permutations, and A or B homogeneous. [Dennunzio et al 2024]

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4) Are there prime X? $\rightarrow$ Is primality decidable?

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Division problem for permutations

instance = couple (A, B) of permutations; its size is $|A + B|$

solution = permutation X such that $AX = B$

$\mathrm{Sol}(A, B)$ = set of solutions

$\mathrm{sol}(A, B)$ = number of solutions

- decision: complexity of deciding if a solution exists $(A \mid B)$?
- counting: complexity of computing the nb of solutions?
- enumeration: complexity of enumerating the solutions?

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Proposition
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A solution X is a permutation of size $n = |B|/|A|$.

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$$2C_2 + C_3 + 3C_5 \equiv 2, 2, 3, 5, 5, 5 \quad \text{(partition of } 22\text{)}$$

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Annoying situation: no better algo, even to decide if $A \mid B$!!!

Product in more details

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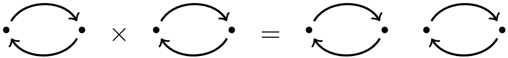

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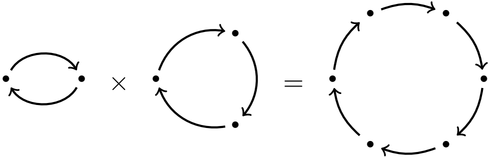

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The cross-lcm between L_A and L_X is in L_B:

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Support of an instance

The support of an instance (A, B) is the largest set $L_{A,B}$ satisfying

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Brut force approach on the support

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Main result

Theorem We can compute the number of solutions X to $AX = B$ in

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Two cases:

- (A, B) basic: $L_{A,B} \subseteq \mathrm{Div}(\mathrm{lcm}\, L_A)$ $\rightarrow$ brut force approach
- (A, B) non-basic $\rightarrow$ divide-and-conquer technique
  - split the instance (A, B) into few basic instances (A_i, B_i),
  - compute the nb of solutions s_i of (A_i, B_i) as in the fist case,
  - output the product of the s_i.

Instance split

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Important property: If $L_{A,B_1} \cap L_{A,B_2} = \emptyset$ then we have a perfect split:

$$\mathrm{Sol}(A, B_1) + \mathrm{Sol}(A, B_2) = \mathrm{Sol}(A, B)$$
$$\mathrm{sol}(A, B_1) \cdot \mathrm{sol}(A, B_2) = \mathrm{sol}(A, B)$$

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$$A = C_2 \qquad \begin{aligned} B &= 2C_2 + 2C_{10} \\ B_1 &= 2C_2 \\ B_2 &= 2C_{10} \end{aligned} \qquad \begin{aligned} L_{A,B} &= \{1, 2, 5, 10\} \\ L_{A,B_1} &= \{1, 2\} \\ L_{A,B_2} &= \{5, 10\} \end{aligned}$$

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$\mathrm{Sol}(A, B) = \mathrm{Sol}(A, B_1) + \mathrm{Sol}(A, B_2)$

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Lemma If (A, B) is non-basic and $\gcd L_{A,B} = 1$, then (A, B) has a perfect split, which can be computed in $O(|A||B|)$.

- If $L_{A,B} \not\subseteq \mathrm{Div}(\mathrm{lcm}\, L_A)$ then there is a prime power p^α in the factorization of $\mathrm{lcm}\, L_B$ such that $p^\alpha \nmid \mathrm{lcm}\, L_A$.
- Let $B = B_1 + B_2$ where
  - B_1 contains the the cycles of B of length kp^α,
    $\hookrightarrow$ $B_1 \neq \emptyset$ since p^α appears in the factorization of $\mathrm{lcm}\, L_B$.
  - B_2 contains the other cycles of B.
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- p^α divides each member of L_{A,B_1} and no member of L_{A,B_2}.
  $\hookrightarrow$ $(A, B_1), (A, B_2)$ is a perfect split.

Summary

- (A, B) basic $\rightarrow$ brut force approach on the support
- (A, B) non-basic and $\gcd L_{A,B} = 1$ $\rightarrow$ perfect split

Instance reduction

Lemma Let (A, B) and $\ell = \gcd L_{A,B}$. Let (A', B') with

- A' obtained from A by replacing each $C_{k\ell}$ by ℓC_k
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Then

- $\mathrm{sol}(A, B) = \mathrm{sol}(A', B')$
- $\operatorname{lcm} L_{A'} \mid \operatorname{lcm} L_A$
- $\gcd L_{A',B'} = 1$.

Summary

- (A, B) basic $\rightarrow$ brut force approach on the support
- (A, B) non-basic $\rightarrow$ reduction $\rightarrow$ perfect split

Main result

Theorem We can compute the number of solutions X to $AX = B$ in

$$O\left(|A||B|^2\left(\frac{|B|}{|A|}\right)^{\mathrm{div}(\mathrm{lcm} L_A)}\right).$$

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Repeating reduction/split, we obtain in $O(|A||B|^2)$ a list of basic instances $(A_1, B_1), \ldots, (A_k, B_k)$ such that

1. $|A_i| = |A|$
2. $\mathrm{lcm}\, L_{A_i} \mid \mathrm{lcm}\, L_A$
3. $|B_1| + \cdots + |B_k| \leq |B|$
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4. $\mathrm{sol}(A, B) = \prod_{i=1}^{k} \mathrm{sol}(A_i, B_i)$.

The brut force approach on the support computes $\mathrm{sol}(A_i, B_i)$ in

$$O\left(|A_i||B_i| \left(\frac{|B_i|}{|A_i|}\right)^{\mathrm{div}(\mathrm{lcm} L_{A_i})}\right) = O\left(|A||B| \left(\frac{|B|}{|A|}\right)^{\mathrm{div}(\mathrm{lcm} L_A)}\right)$$

Conclusion and Perspectives

Given two functional digraphs A, B, complexity of deciding if $A \mid B$?

Polynomial when:

- B is a dendron. [Naquin, Gadouleau 2024]
- A, B are permutations, and A or B homogeneous [Dennunzio et al 2024+]
- A, B are permutations, A fixed. [this talk]
- A is a fixed permutation (by combining items 1 and 3). [unpublished]

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Perspectives:

- Reduce the general case to permutations. [Marius Rolland]
- Polynomial algorithm for any fixed A.
- Beat the brut force algorithm for permutations A, B, running in

$$e^{O(\sqrt{|B|/|A|})}.$$