Synchronizing Boolean networks asynchronously

work in progress

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joint work with

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Outline

1. **Synchronizing Deterministic Finite Automata**
2. Synchronizing Boolean Networks
3. Conclusion

A **Deterministic Finite Automaton (DFA)** consists of

- a finite alphabet A,
- a finite set of states Q,
- for each letter $a \in A$, a function $f^a : Q \to Q$.

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A DFA is **synchronizing** if it has a synchronizing word.

Example with $A = \{a, b\}$, $Q = \{1, 2, 3, 4\}$ and

x	$f^a(x)$	$f^b(x)$
1	2	2
2	3	2
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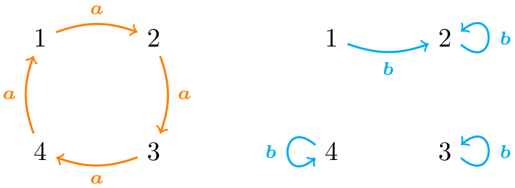

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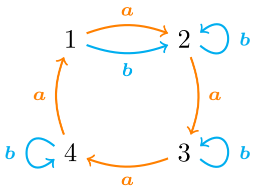
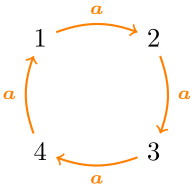
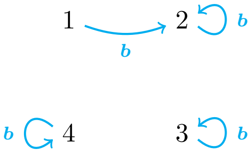


State Transition Graph





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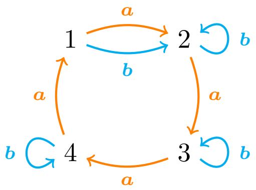

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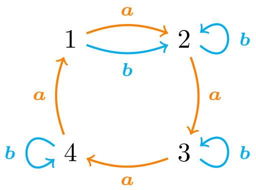

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1

2

3

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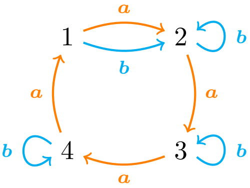

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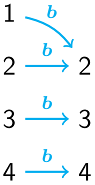

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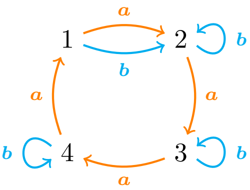

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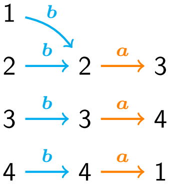

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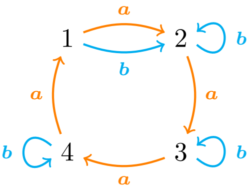

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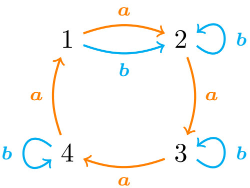

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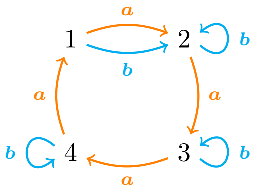

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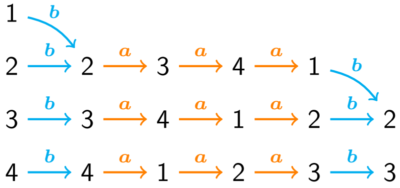

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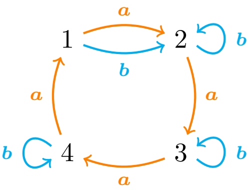

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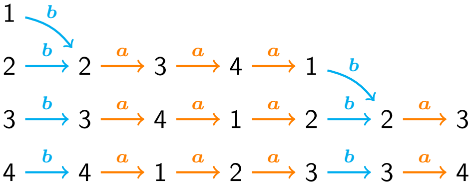

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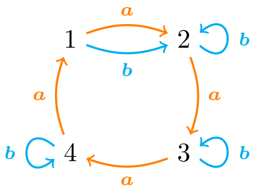

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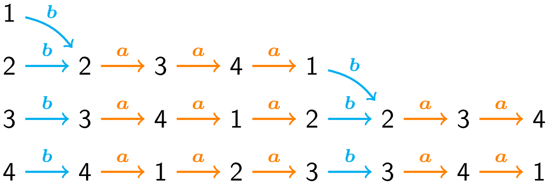

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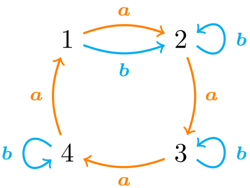

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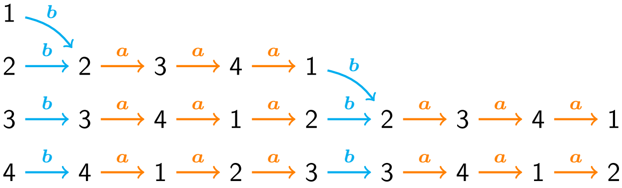

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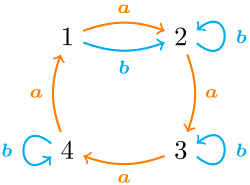

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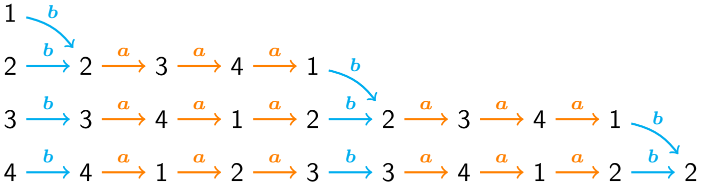

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Road Coloring Theorem [Conjectured in 70, proved by Trahtman 08]

Let D be a strong digraph with **loop number one**, where each vertex has n out-going arcs (with possibly identical ends).

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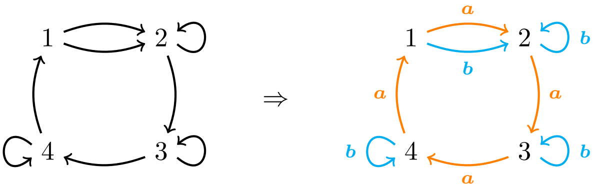

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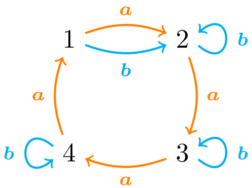

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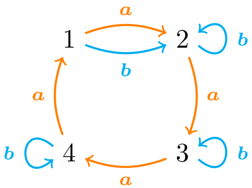

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And he give the following useful observation.

Lemma. A DFA is synchronizing iff, for any two states x, y, there is w s.t.

$$f^w(x) = f^w(y).$$

Theorem

If a DFA with q states is synchronizing, then it has synchronizing word of length at most

$\frac{1}{2} \cdot q(q-1)^2$	[Starke 66]
$\frac{1}{6} \cdot (q^3 - q)$	[Frankl 82, Pin 82]
$\frac{4409}{4410} \cdot \frac{1}{6} \cdot q^3 + O(q^2)$	[Skykula 18]

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Weak Černý's Conjecture

If a DFA with q states is synchronizing, then it has synchronizing word of length at most $O(q^2)$.

Outline

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2. **Synchronizing Boolean Networks**
3. Conclusion

A **Boolean Network (BN)** with n components is a function

$$\begin{aligned} f : \{0,1\}^n &\to \{0,1\}^n \\ x = (x_1, \ldots, x_n) &\mapsto f(x) = (f_1(x), \ldots, f_n(x)) \end{aligned}$$

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Global transition function

Locale transition functions
(from $\{0,1\}^n$ to $\{0,1\}$)

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Asynchronous dynamics: **one** component is updated at each step.

$\hookrightarrow$ Classical model for **gene networks** [Thomas 1969].

$\hookrightarrow$ Update component i at state x means reach the state

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The **associated DFA** is defined by

- The alphabet is $A = \{1, \dots, n\}$.
- The set of states is $Q = \{0,1\}^n$.
- The function associated to each $i \in A$ is $f^i : \{0,1\}^n \to \{0,1\}^n$.

Local transition functions

$$\begin{cases} f_1(x) = \overline{x_1} \wedge \overline{x_2} \\ f_2(x) = x_1 \end{cases}$$

Global transition function

x	$f(x)$
00	10
01	10
10	11
11	01

Associated DFA

x	$f^{1}(x)$	$f^{2}(x)$
00	10	00
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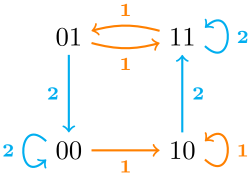

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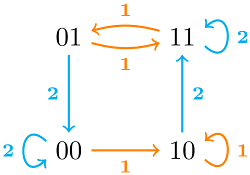
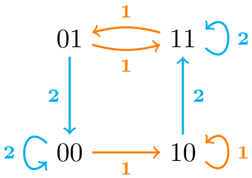
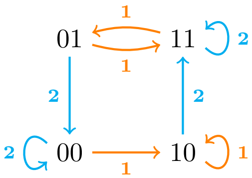


Asynchronous State Transition Graph



$w = 2112$ is synchronizing



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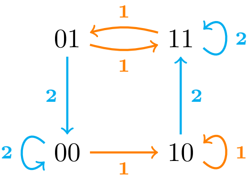
00

01

10

11



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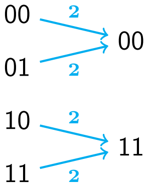

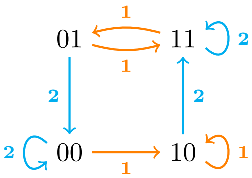

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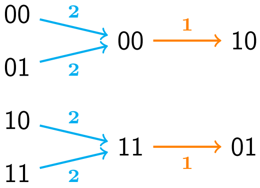

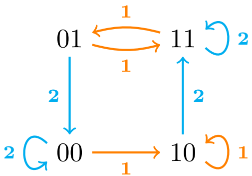

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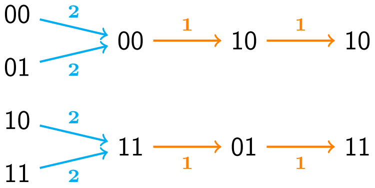

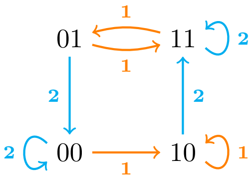

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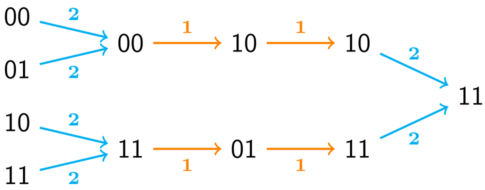

The **interaction graph (IG)** of f is the **signed digraph** G defined by

- the vertex set is $\{1, \ldots, n\}$,

- there is a positive edge $\boldsymbol{j} \rightarrow \boldsymbol{i}$ if there is $x \in \{0,1\}^n$ such that

$$f_i(x_1, \ldots, x_{j-1}, \mathbf{0}, x_{j+1}, \ldots, x_n) = \mathbf{0}$$
$$f_i(x_1, \ldots, x_{j-1}, \mathbf{1}, x_{j+1}, \ldots, x_n) = \mathbf{1}$$

- there is a negative edge $\boldsymbol{j} \rightarrow \boldsymbol{i}$ if there is $x \in \{0,1\}^n$ such that

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$$f_i(x_1, \ldots, x_{j-1}, \mathbf{1}, x_{j+1}, \ldots, x_n) = \mathbf{0}$$

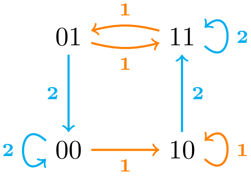
Local transition functions

$$\begin{cases} f_1(x) = \overline{x_1} \wedge \overline{x_2} \\ f_2(x) = x_1 \end{cases}$$

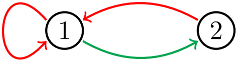
Global transition function

x	$f(x)$
00	00
01	10
10	01
11	01

Asynchronous State Transition Graph



Interaction graph



Definitions

- A BN f is **synchronizing** if its associated DFA is.
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Questions

- Which BNs f are synchronizing?
- Which interaction graphs G are synchronizing?
- Is Černý's conjecture true for BNs?
- Is Černý's conjecture true for BNs with a synchronizing IG?

Remark 1 If G is **acyclic**, then G is synchronizing.

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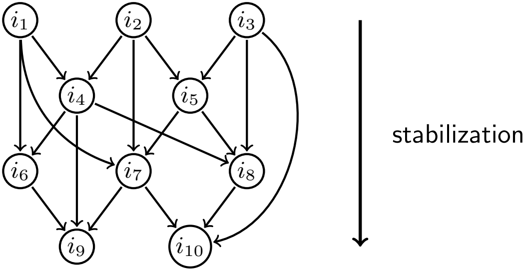
If G is acyclic then we know that any BN f has a unique fixed point x.
If $\boldsymbol{w} = i_1 i_2 \dots i_n$ is a **topological sort**, then $\boldsymbol{w}$ is a synchronizing word:

$$f^w = \mathrm{cst} = x.$$



Remark 2 If G is synchronizing, then it has a vertex of in-degree 0 or 2.

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More precisely, if G has no vertex of in-degree 0 or 2, then there is a BN f on G which is **self-dual**, that is, for any x,

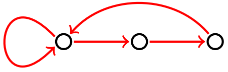
$$f(x) = \overline{f(\overline{x})},$$

and then, for any word w,

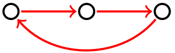
$$f^w(x) = \overline{f^w(\overline{x})}.$$

Remark 3

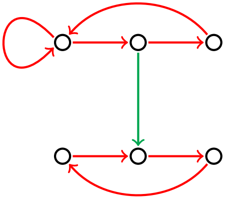
- If G is synchronizing, then all its initial strong components are, but some non-initial strong components can be non-synchronizing.



Synchronizing



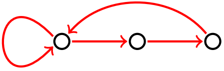
Not synchronizing



Synchronizing

Remark 3

- If G is synchronizing, then all its initial strong components are, but some non-initial strong components can be non-synchronizing.
- If all the strong components of G are synchronizing, then G is not necessarily synchronizing.



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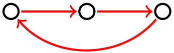

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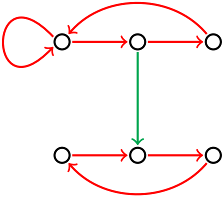

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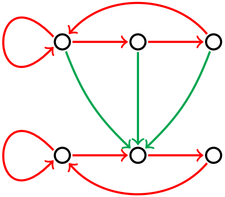

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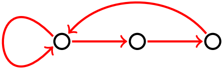

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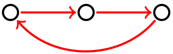

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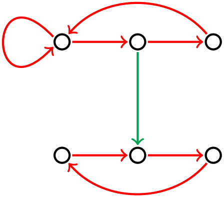

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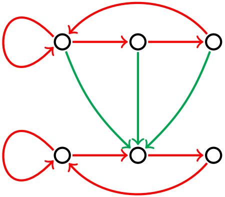

Not synchronizing

$\hookrightarrow$ It is natural to focus on **strongly connected** interaction graphs.

Remark 4 If G is strong and synchronizing, it has a **negative cycle**.

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Theorem [Aracena 08]

If G is strong and has **no negative cycle** then every BN f on G has at least two fixed points (and is thus not synchronizing).

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If G is strong and has **no negative cycle** then every BN f on G has at least two fixed points (and is thus not synchronizing).

Remark 5 If a DFA has multiple terminal strong components, then it is **not** synchronizing.

Theorem [Comet and R. 07]

If G has **only negative cycles**, then the DFA associated with every BN f on G has a **unique** terminal strong component.

$\hookrightarrow$ It is natural to focus on strong IGs with **only negative cycles**.

Theorem 1

Suppose that G has the following three properties:

(1) G is **strong**,

(2) G has **only negative cycles**,

(3) G has **max in-degree 2**.

Then every BN on G has a synchronizing word of length $O(2^{2n})$.

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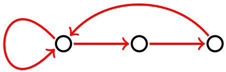

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Theorem 2

- If G has properties (2) and (3): **coNP-hard** to decide if G is synch.
- If G has properties (1) and (3): **coNP-hard** to decide if G is synch.
- If G has properties (1) and (2): G is not necessarily synchronizing.

An **and-or-net** is a BN f such that each local function f_i is

- a conjunction of positive or negative literals, or
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Theorem 1'

Suppose that G has the following three properties:

(1) G is **strong**,

(2) G has **only negative cycles**,

(3) G is **not a cycle**,

Then every **and-or-net** on G has a synchronizing word of length $O(2^{2n})$.

Definition: G is **good** if it has only negative cycles and, for every vertex i and every initial strong component S not containing i, there is $j \in S$ which is **homogenous** for i, that is, all the paths from j to i have the same sign.

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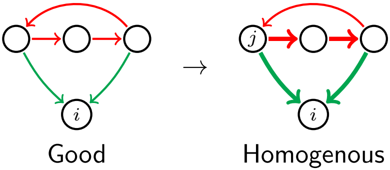

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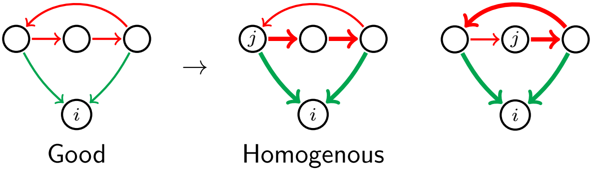

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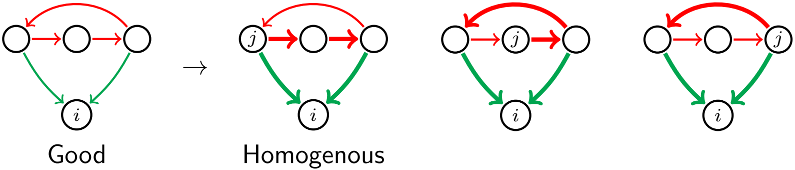

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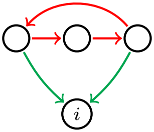
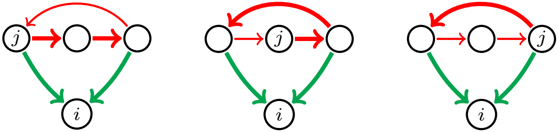
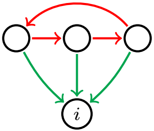


Good

$\rightarrow$



Homogenous



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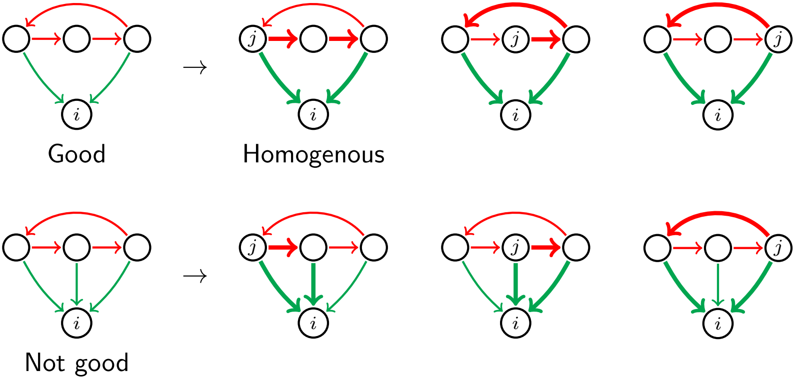

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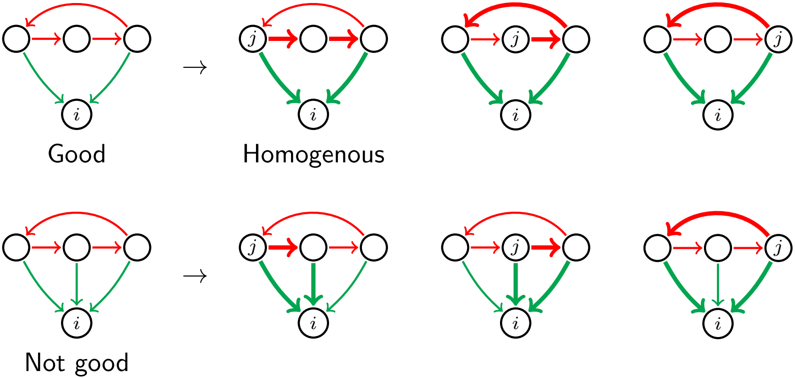
Remark: If G is **strong** and has **only negative cycles**, it is **good**.



Main result

If G is **good**, has **no source** and **is not a cycle**, then every and-or-net on G has a synchronizing word of length $O(2^{2n})$.

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Suppose that G is **good** and has **no source**. Let f be a BN on G.
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Let us say that G is **and-or-synchronizing** if every and-or-net on G is.

Lemma 2

Suppose that G is **good** and has **no source**. If each strongly connected component of G is and-or-synchronizing, then G and-or-synchronizing.

Outline

1. Synchronizing Deterministic Finite Automata
2. Synchronizing Boolean Networks
3. **Conclusion**

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Many open questions:

- Černý's Conjecture for BNs.
- Černý's Conjecture for BNs with a synchronizing interaction graphs.
- Which interaction graphs admit at least one synchronizing BN?

Gracias!