Solving Numeric Constraints

Michel RUEHER

CeP Project I3S–CNRS

Université de Nice–Sophia Antipolis

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Overview

- ## Interval Programming
  - Interval arithmetic
  - Interval Analysis methods

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  - Interval arithmetic
  - Interval Analysis methods

- ## Constraint Programming
  - Overall scheme
  - Local consistencies
  - Quantified constraints
  - Global Constraints

1. Interval Programming

- $\longrightarrow$ Basics on interval arithmetics
- $\longrightarrow$ Interval Newton-like methods for solving a multi-variate system of non-linear equations

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- $\diamond$ $\mathbf{x}, \mathbf{y}$: real variables or vectors; $\mathbf{X}, \mathbf{Y}$ interval variables or vectors

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Let f be a real-valued function of n unknowns $\mathbf{X} = \{x_1, \ldots, x_n\}$, an **interval evaluation $\mathbf{F}$** of f for given ranges $\{X_1, \ldots, X_n\}$ for the unknowns is an interval Y such that

$$\forall \mathbf{X} = \{\mathbf{x_1}, \ldots, \mathbf{x_n}\} \in \mathcal{X} = \{\mathbf{X_1}, \ldots, \mathbf{X_n}\} \qquad \underline{\mathbf{Y}} \leq \mathbf{f}(\mathbf{X}) \leq \overline{\mathbf{Y}}$$

$\underline{\mathbf{Y}}, \overline{\mathbf{Y}}$ are lower and upper bounds for the values of f when the values of the unknowns are restricted to the box $\mathcal{X}$

1.1 Interval arithmetics : basic definitions (2)

A relation over the intervals $C : \mathcal{I}^n \longrightarrow \mathcal{B}ool$ is an interval extension of the relation $c : \mathcal{R}^n \longrightarrow \mathcal{B}ool$ iff:

$$\forall \mathbf{I_1}, \ldots, \mathbf{I_n} \in \mathcal{I} : \mathbf{r_1} \in \mathbf{I_1}, \ldots, \mathbf{r_n} \in \mathbf{I_n} \;\&\; \mathbf{c}(\mathbf{r_1}, \ldots, \mathbf{r_n}) \Rightarrow \mathbf{C}(\mathbf{I_1}, \ldots, \mathbf{I_n})$$

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For instance, $\mathbf{I_1} \doteq \mathbf{I_2} \Leftrightarrow (\mathbf{I_1} \cap \mathbf{I_2}) \neq \emptyset$ is an interval extension of the relation $=$ over the real numbers

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→ The interval extension of a function is an interval function that computes an **outer approximation** of the range of the function over a domain

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- $[\mathbf{a}, \mathbf{b}] \otimes [\mathbf{c}, \mathbf{d}] = [\mathrm{min}(\mathbf{ac}, \mathbf{ad}, \mathbf{bc}, \mathbf{bd}), \mathrm{max}(\mathbf{ac}, \mathbf{ad}, \mathbf{bc}, \mathbf{bd})]$

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- $[\mathbf{a},\mathbf{b}] \oslash [\mathbf{c},\mathbf{d}] = [\min(\frac{\mathbf{a}}{\mathbf{c}},\frac{\mathbf{a}}{\mathbf{d}},\frac{\mathbf{b}}{\mathbf{c}},\frac{\mathbf{b}}{\mathbf{d}}),\max(\frac{\mathbf{a}}{\mathbf{c}},\frac{\mathbf{a}}{\mathbf{d}},\frac{\mathbf{b}}{\mathbf{c}},\frac{\mathbf{b}}{\mathbf{d}})]$ ***if*** $\mathbf{0} \notin [\mathbf{c},\mathbf{d}]$

Interval extension : properties

$\longrightarrow$ If $0 \notin F(\mathcal{X})$, then no value exists in the box $\mathcal{X}$ such that $f(\mathbf{X}) = 0$
$\Leftrightarrow$ **the equation $f(\mathbf{X})$ has no root in the box $\mathcal{X}$**

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but interval arithmetics is **sub-distributive**:

$$\mathbf{X}(\mathbf{Y} + \mathbf{X}) \subseteq \mathbf{XY} + \mathbf{XZ}$$

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Example :

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$\mathbf{X(X - 1) = [0, 5]([0, 5] - [1, 1]) = [0, 5][-1, 4] = [-5, 20]}$

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$\longrightarrow$ First-order Taylor development of $\mathbf{f}$

$$\mathbf{F}_{\mathsf{tay}}(\mathbf{X}) = \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{X}).(\mathbf{X} - \mathbf{x})$$

with $\forall \mathbf{x} \in \mathbf{X}$, $\mathbf{J}()$ being the Jacobian of $\mathbf{f}$

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$$\mathbf{f}([\mathbf{0}, \mathbf{2}]) = \mathbf{1} - \mathbf{X} + \mathbf{X}^{\mathbf{2}} = \mathbf{1} - [\mathbf{0}, \mathbf{2}] + [\mathbf{0}, \mathbf{2}]^{\mathbf{2}} = [-\mathbf{1}, \mathbf{5}]$$

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$$\mathbf{f}_{\mathsf{factor}}([\mathbf{0}, \mathbf{2}]) = \mathbf{1} + \mathbf{X}(\mathbf{X} - \mathbf{1}) = \mathbf{1} + [\mathbf{0}, \mathbf{2}]([\mathbf{0}, \mathbf{2}] - \mathbf{1}) = [-\mathbf{1}, \mathbf{3}]$$

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$$\mathbf{f}_{\mathsf{tay}}([\mathbf{0}, \mathbf{2}]) = \mathbf{f(x) + (2X - 1)(X - x)}$$
$$= \mathbf{f(1) + (2[0, 2] - 1)([0, 2] - 1)} = [\mathbf{-2}, \mathbf{4}]$$

$$\mathbf{f}([\mathbf{0}, \mathbf{2}]) = \mathbf{1 - X + X^2 = 1 - [0, 2] + [0, 2]^2} = [\mathbf{-1}, \mathbf{5}]$$

$$\mathbf{f}_{\mathsf{factor}}([\mathbf{0}, \mathbf{2}]) = \mathbf{1 + X(X - 1) = 1 + [0, 2]([0, 2] - 1)} = [\mathbf{-1}, \mathbf{3}]$$

whereas the range of f over $\mathbf{X} = [\mathbf{0}, \mathbf{2}]$ is $[\mathbf{3/4}, \mathbf{3}]$

1.2 Interval Analysis methods

Goal : to determine the zeros of a system of n equations $f_i(x_1,\ldots,x_n)$ in n unknowns x_i inside the interval vector $X=\{X_1,\ldots,X_n\}$ with $x_i \in X_i$ for $i=1,\ldots,n$

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Consider the case of interval linear equations :

$$\mathbf{A}.\mathbf{x} = \mathbf{b}$$

with $\mathbf{A}$ an interval matrix and $\mathbf{b}$ an interval vector

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For each unknowns X_i, the Gauss-Seidel algorithm is defined by the following iterative process:

$$\mathbf{X_i^{k+1} = (b_i - \sum_{j=1}^{i-1} A_{i,j} X_j^{k+1} - \sum_{j=i+1}^{n} A_{i,j} X_j^{k}) / A_{i,i} \cap X_i^k}$$

Pre-conditioning $\longrightarrow$ to shrink the width of the intervals

Interval Newton algorithm (1)

Principle of the Newton operator :

Consider $\mathbf{f} : \mathcal{R} \longrightarrow \mathcal{R}$, the mean value theorem says :

$\exists \mathbf{a} \in [\mathbf{v}, \mathbf{u}] : \mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{v}) = (\mathbf{u} - \mathbf{v})\mathbf{f}'(\mathbf{a})$ and thus,
$\mathbf{f} \ : \ \mathbf{v} = \mathbf{u} - \frac{\mathbf{f}(\mathbf{u})}{\mathbf{f}'(\mathbf{a})}$ if $\mathbf{v}$ is a zero of $\mathbf{f}$

If $\mathbf{a} \in \mathbf{I}$ then $\mathbf{f}(\mathbf{a}) \in \mathbf{F}(\mathbf{I})$, and $\mathbf{v} \in \tilde{\mathbf{u}} - \frac{\mathbf{F}(\tilde{\mathbf{u}})}{\mathbf{F}'(\mathbf{I})} = \mathbf{N}(\mathbf{F}, \mathbf{F}', \tilde{\mathbf{u}}, \mathbf{I})$

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If $\mathbf{v}$ is a zero of $\mathbf{f}$ then $\mathbf{v} \in \mathbf{I_n} \quad (\mathbf{n} \geq \mathbf{1})$ where

$$\mathbf{I_0} = \mathbf{I}$$
$$\mathbf{I_{i+1}} = \mathbf{N}(\mathbf{F}, \mathbf{F}', \mathbf{center}(\mathbf{I_i}), \mathbf{I}) \cap \mathbf{I_i}$$
$$\mathbf{I_n} = \mathbf{I_{n+1}}$$

Interval Newton algorithm (2)

The Interval Newton algorithm is used to solve **non-linear systems** with

$$\mathbf{X_{k+1} = N(\tilde{x}_k, X_k) \cap X_k} \quad \text{with} \quad \mathbf{N(\tilde{x}_k, X_k) = \tilde{x}_k - A.f(\tilde{x}_k)}$$

where $\mathbf{A = [F^{'}(X_k)]^{-1}}$ and $\mathbf{\tilde{x}_k \in X_k}$ (e.g., the mid-point of $\mathbf{X_k}$)

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where $\mathbf{A} = [\mathbf{F}^{'}(\mathbf{X_k})]^{-1}$ and $\tilde{\mathbf{x}}_\mathbf{k} \in \mathbf{X_k}$ (e.g., the mid-point of $\mathbf{X_k}$)

Properties :

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Matrix $\mathbf{A} = [\mathbf{F}^{'}(\mathbf{X_k})]^{-1}$ may be costly to compute
… to determine $N(\tilde{x}_k, X_k) \rightarrow$ solve the linear system:

$$\mathbf{F}^{'}(\mathbf{X_k})(\mathbf{N}(\mathbf{\tilde{x}_k}, \mathbf{X_k}) - \mathbf{\tilde{x}_k}) = -\mathbf{f}(\mathbf{\tilde{x}_k})$$

2. Constraint Programming

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(D_{x_i} contains all acceptable values for variable x_i)

◇ $\mathcal{C} = \{c_1, \ldots, c_m\}$ is a set of constraints

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2. **Making a choice to generate two (or more) sub-problems**

◇ The pruning step $\longrightarrow$ **reduces an interval** when it can prove that the upper bound or the lower bound does not satisfy some constraint

◇ The branching step $\longrightarrow$ **splits the interval** associated to some variable in two intervals (often with the same width)

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Consider $\mathbf{X} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and $\mathbf{C}(\mathbf{x}, \mathbf{x_1}, \ldots, \mathbf{x_n}) \in \mathcal{C}$: if $\mathbf{C}(\mathbf{x}, \mathbf{x_1}, \ldots, \mathbf{x_n})$ does not hold for any values $\mathbf{a} \in [\underline{\mathbf{x}}, \mathbf{x}']$, then X may be shrinked to $\mathbf{X} = [\mathbf{x}', \overline{\mathbf{x}}]$

2.2 Local consistencies (1)

→ Informally speaking, a constraint system C satisfies a partial consistency property if **a relaxation of C is consistent**

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→ A **constraint C_j is AC-like-consistent** if for any variable x_i in $\mathcal{X}_j$, **the bounds $\underline{D_i}$ and $\overline{D_i}$ have a support** in the domains of all other variables of $\mathcal{X}_j$

Local consistencies used in `BNR-prolog`, `Interlog`, `CLP(BNR)`, `PrologIV`, `UniCalc`, `Ilog Solver`, `Numerica`, `Icos`, `RealPaver` are AC-like-consistencies

2.2 Local consistencies (2)

Local consistencies are conditions that filtering algorithms must satisfy

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$\longrightarrow$ fixed point algorithm defined by the sequence $\{\mathcal{D}_k\}$ of domains generated by the iterative application of an operator

$$\mathbf{Op} : \mathbb{I}(\mathbb{R})^{\mathbf{n}} \longrightarrow \mathbb{I}(\mathbb{R})^{\mathbf{n}}$$

$$\boxed{\mathcal{D}_k = \begin{cases} \mathcal{D} & \text{if } k = 0 \\ Op(\mathcal{D}_{k-1}) & \text{if } k > 0 \end{cases}}$$

2.2 Local consistencies (3)

Properties of the operator Op :

$\diamond$ $\mathbf{Op}(\mathcal{D}) \subseteq \mathcal{D}$ (contractance)

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The limit of the sequence $\{\mathcal{D}_k\}$, which corresponds to the greatest fixed point of the operator $\mathbf{O}p$

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→ **Box–consistency** is a coarser relaxation of Arc–Consistency than 2B–consistency ... **but Box–consistency algorithms actually achieve a stronger filtering than 2B–consistency**

→ **Strong consistencies** : no bound of the domains can be removed with a local consistency filtering algorithm

2B–consistency (1)

Variable x is 2B–consistency for constraint $\mathbf{f(x, x_1, \ldots, x_n) = 0}$ if the lower (resp. upper) bound of the domain $\mathbf{X}$ is the smallest (resp. largest) solution of $\mathbf{f(x, x_1, \ldots, x_n)}$

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Definition : 2B–consistency

Let $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ be a CSP and $\mathbf{C} \in \mathcal{C}$ a k-ary constraint over $(\mathbf{X_1, \ldots, X_k})$

$\mathbf{C}$ is 2B–consistency iff :

$$\forall \mathbf{i}, \mathbf{X_i} = \Box\{\mathbf{\tilde{x}_i} \mid \exists \mathbf{\tilde{x}_1} \in \mathbf{X_1}, \ldots, \exists \mathbf{\tilde{x}_{i-1}} \in \mathbf{X_{i-1}}, \exists \mathbf{\tilde{x}_{i+1}} \in \mathbf{X_{i+1}}, \ldots, \exists \mathbf{\tilde{x}_k} \in \mathbf{X_k} \text{ such that } \mathbf{c}(\mathbf{\tilde{x}_1}, \ldots, \mathbf{\tilde{x}_{i-1}}, \mathbf{\tilde{x}_i}, \mathbf{\tilde{x}_{i+1}} \ldots, \mathbf{\tilde{x}_k}) \text{ holds}\}$$

A CSP is 2B–consistency iff all its constraints are 2B–consistency

Box–consistency (1)

Variable $\mathbf{x}$ is Box–Consistent for constraint $\mathbf{f(x, x_1, \ldots, x_n) = 0}$ if the bounds of the domain of x correspond to the leftmost and the rightmost zero of the optimal interval extension of $\mathbf{f(x, x_1, \ldots, x_n)}$

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$\mathbf{C}$ is Box–Consistent if, for all $\mathbf{X_i}$ the following relations hold :

1. $\mathbf{C(X_1, \ldots, X_{i-1}, [\underline{X_i}, \underline{X_i}^{+}), X_{i+1}, \ldots, X_k)}$
2. $\mathbf{C(X_1, \ldots, X_{i-1}, (\overline{X_i}^{-}, \overline{X_i}], X_{i+1}, \ldots, X_k)}$

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- For complex constraints, no analytic solution function may exist Consider $\mathbf{x + \log(x) = 0}$

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   ◇ This approach is used for computing **2B–consistency filtering** (the initial constraints are decomposed into primitive constraints)

   ◇ Decomposition does not change the semantics of the initial constraints system but **it amplifies the dependency problem**

Local consistency filtering (3)

2. Transformation of the constraint $\mathbf{C_j(x_{j_1}, \ldots x_{j_k})}$ into $\mathbf{k}$ mono-variable constraints $\mathbf{C_{j,l}, l = 1 \ldots k}$ by substituting their intervals for the other variables

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$\longrightarrow$ The two extremal zeros of $\mathbf{C_{j,l}}$ can be found by a **dichotomy algorithm combined with a mono-variable version of the interval Newton method**

$\longrightarrow$ This approach is **well adapted for Box–consistency filtering**

Local consistency filtering (4)

3. Use the Taylor extension to transform the constraint into an interval linear constraint. $f(X) = 0$ becomes

$$\mathbf{f(c)} + \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{nat}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x_i}}\right)(\mathbf{X}) * (\mathbf{X_i} - \mathbf{c_i}) = \mathbf{0}$$

   where $\mathbf{c} = \mathbf{m}(\mathbf{X})$. The derivatives are evaluated over a box $\mathcal{D}$ that contains $\mathbf{X}$, $\mathcal{D}$ is considered as constant, and with $\mathbf{c} = \mathbf{m}(\mathcal{D})$

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$\longrightarrow$ **Solving the squared interval linear system allows much more precise approximations of projections to be computed**

Stronger consistencies, 3B–Consistency (1)

3B–Consistency, a relaxation of path consistency, checks whether 2B–Consistency can be enforced when the domain of a variable is reduced to the value of one of its bounds in the whole system

Stronger consistencies, 3B–Consistency (2)

Definition : 3B–Consistency
Let $(\mathcal{X}, \mathcal{D}, \mathcal{C})$ be a CSP and $\mathbf{x}$ a variable of $\mathcal{X}$ with $\mathbf{D_x} = [\mathbf{a}, \mathbf{b}]$. Let also:

◇ Let $\mathbf{P}_{\mathbf{D^1_x} \leftarrow [\mathbf{a}, \mathbf{a}^+)}$ be the CSP derived from $\mathbf{P}$ by substituting $\mathbf{D_x}$ in $\mathcal{D}$ with $\mathbf{D^1_x} = [\mathbf{a}, \mathbf{a}^+)$

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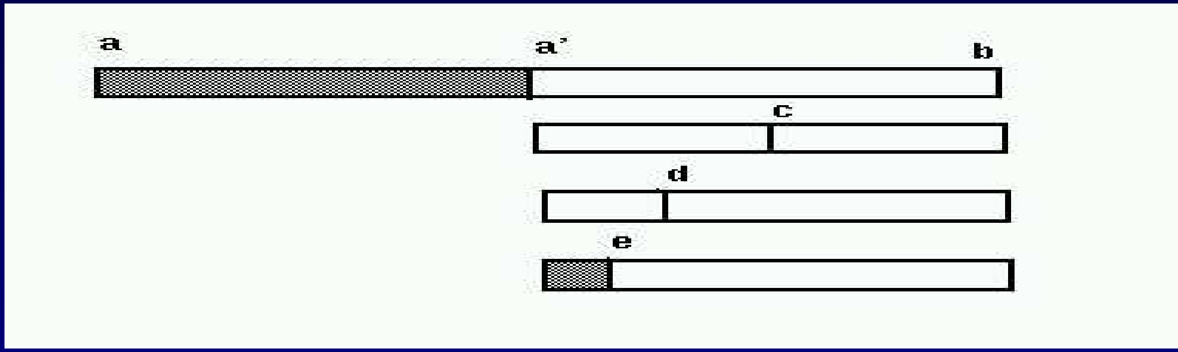
$\mathbf{X}$ is 3B–Consistent iff $\mathbf{\Phi_{2B}}(\mathbf{P}_{\mathbf{x} \leftarrow [\underline{\mathbf{x}}, \underline{\mathbf{x}}^+)}) \neq \mathbf{P}_\emptyset$ and
$\mathbf{\Phi_{2B}}(\mathbf{P}_{\mathbf{x} \leftarrow (\overline{\mathbf{x}}^-, \overline{\mathbf{x}}]}) \neq \mathbf{P}_\emptyset$

Stronger consistencies, 3B–Consistency (3)



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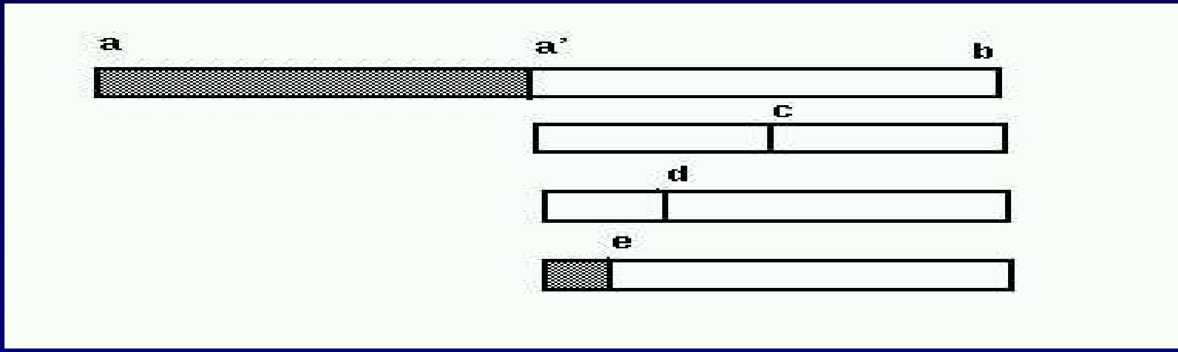

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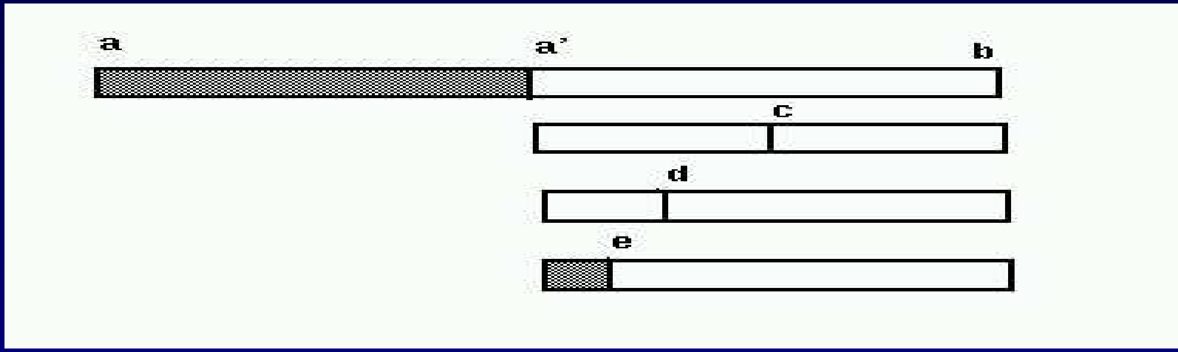

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◇ otherwise, the filtering process continues on the interval $[\mathbf{a}, \frac{3\mathbf{a}+\mathbf{b}}{4}]$.

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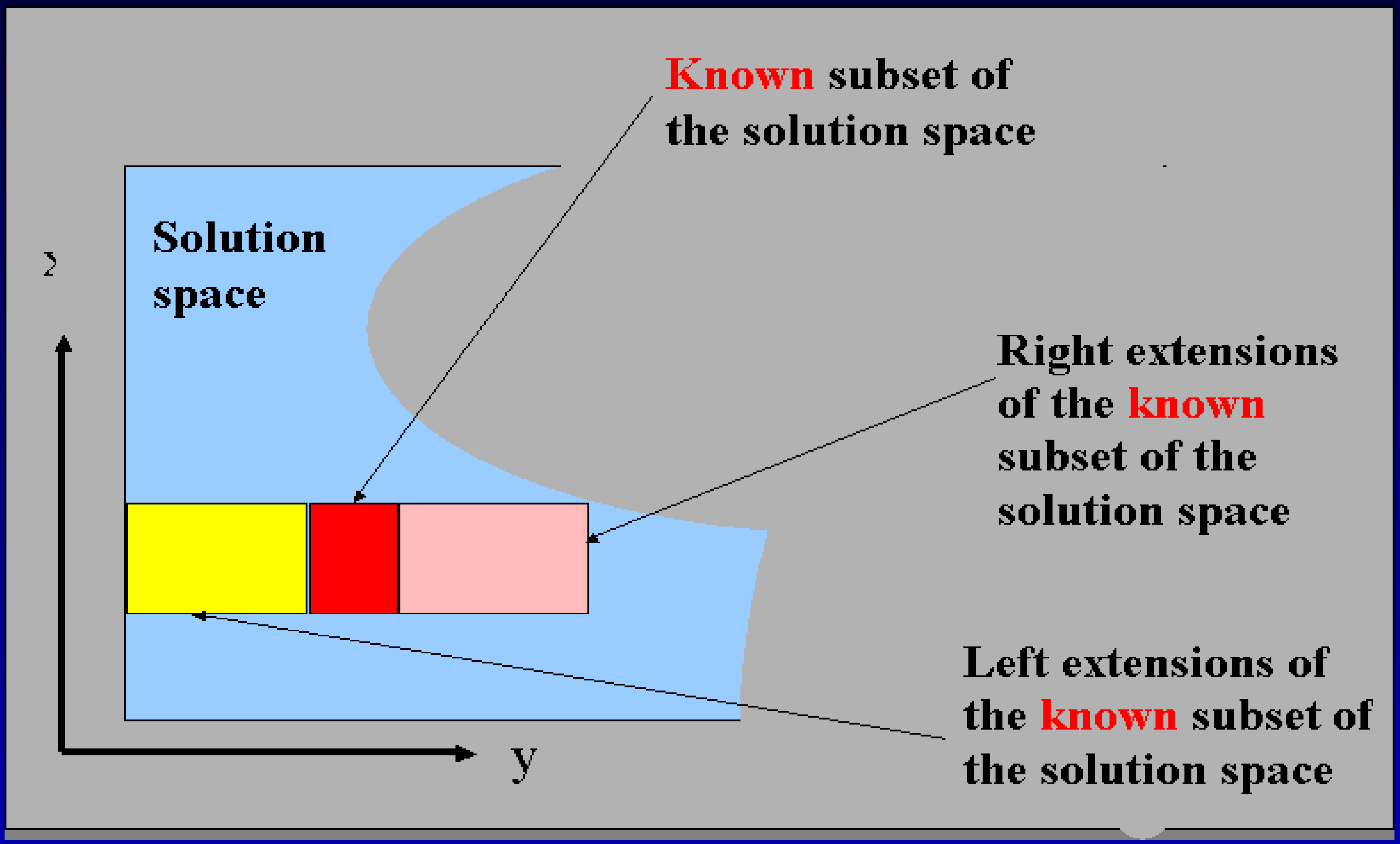

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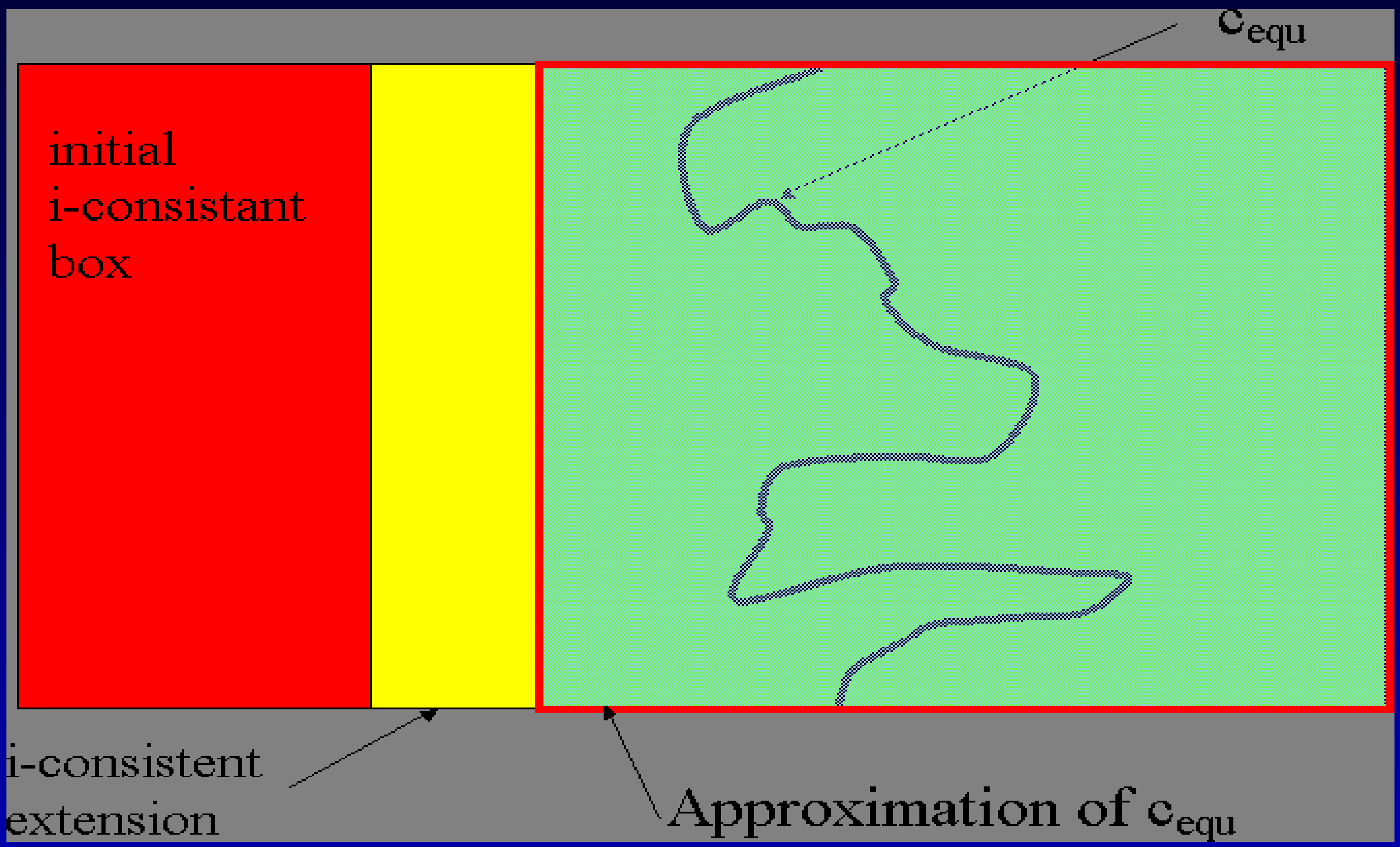

2.3. Quantified constraints (3)

◇ $\forall \mathbf{x} \in \mathbf{D_x} : \ \mathbf{x} + \mathbf{x_1} \ < \mathbf{5} \quad \textbf{with} \quad \mathbf{D_x} = [-\mathbf{2}, \mathbf{2}], \mathbf{D_{x_1}} \ = [\mathbf{1}, \mathbf{5}]$

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◇ Modal intervals

2.4 Global Constraints (1)

◇ **"Syntactical" approach**

To handle an approximation of the whole constraint system with the **simplex algorithm**

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$\longrightarrow$ introduce redundant linear constraints to get a **tight approximation** of the non-linear terms

$\longrightarrow$ solving a linear relaxation with the simplex algorithm

2.4 Global Constraints (2)

◇ **“Semantic” approach**

⟶ Distance constraint