UNIVERSITÉ CÔTE D'AZUR

Modélisation de l'incertitude (M2 MIAGE IA²)

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Séance 7
Théorie des possibilités

Dans cette séance

- Introduction
- Logic and Uncertainty
- Possibility Distribution
- Possibility Measures
- Possibilistic Logic

Introduction

- Uncertainty pervades information and knowledge
- The handling of uncertainty in inference systems has been an issue for a long time in AI
- Logical formalisms have dominated AI for several decades
  - Modal logic
  - Non-monotonic logic
  - Many-valued logic (e.g., fuzzy logic)
- Bayesian networks have become prominent in AI, but...
  - They “deny” the problem of incomplete knowledge
  - The self-duality of probabilities cannot distinguish the lack of belief in a proposition and the belief in its negation

Logic and Uncertainty

- There has been a divorce between logic and probability in the early 20th century
  - Logic as a foundation for Mathematics
  - Probability instrumental to represent statistical data
- Attempts at logical probability have been unsuccessful
  - Degrees of belief cannot be additive and self-dual
  - Deductions lead to incompatible probabilities or, at best, at interval-valued probabilities
- Some theories have emerged to overcome this problem
  - Walley's imprecise probability theory
  - Dempster-Shafer theory of evidence
  - Possibility theory

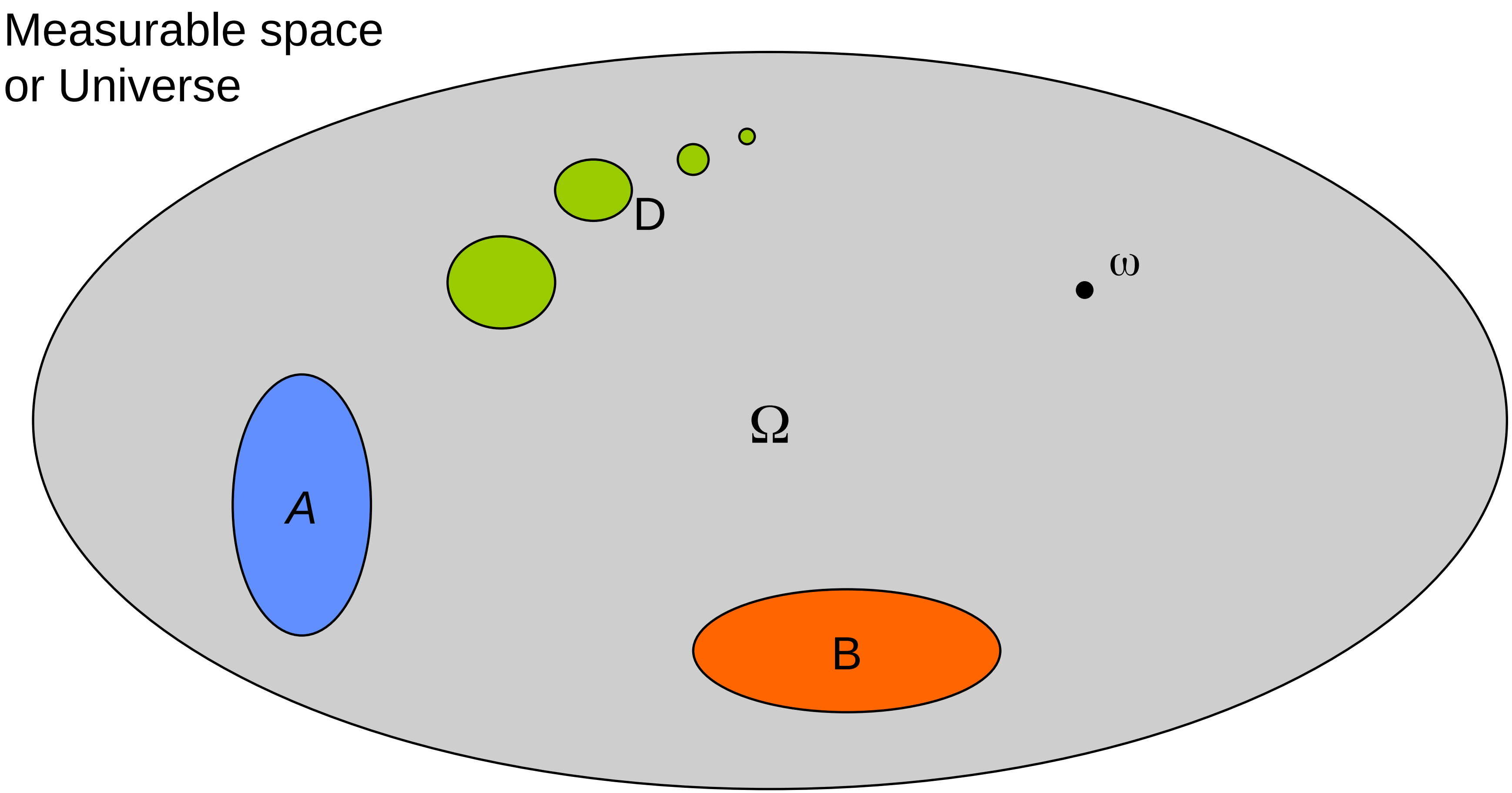
Representing Beliefs

When knowledge is incomplete, we may speak of **Beliefs**

There appears to be three traditions for representing beliefs

- Set-Functions
- Multiple-Valued Logic
- Modal Logic

Interpretations / Events



Set Functions

- A set function is used to assign degrees of beliefs to propositions
- A proposition is represented by the set of its models

But Spohn:

$$f : 2^{\Omega} \to [0, 1] \qquad \kappa : 2^{\Omega} \to \mathbb{N}$$

$$A \subseteq B \Leftrightarrow f(A) \leq f(B)$$

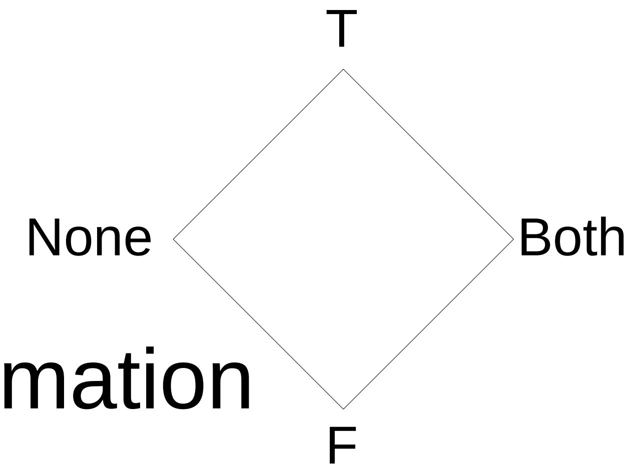
$$f(\emptyset) = 0$$

$$f(\Omega) = 1$$

Probability
Possibility and Necessity
Belief and Plausibility
Upper and Lower Probability

Multiple-Valued Logic

- Three-valued logics (Łukasiewicz, Kleene):
  - True
  - False
  - Possible, Unknown, or Indeterminate
- Four-valued logic (Belnap):
  - T, F, Both, None
  - None means unknown
  - Both means conflicting information



Modal Logic

- Represent the belief modality at the syntactic level
- Necessity symbol (modality) $\Box P$
- Clear distinction between $\neg\Box P \quad \Box\neg P$
- Doxastic Logic (von Wright, Hintikka, Fagin *et al.*)
- Axioms:
  - K $\quad \Box(P \Rightarrow Q) \Rightarrow (\Box P \Rightarrow \Box Q)$
  - D $\quad \Box P \Rightarrow \neg\Box\neg P$
  - T $\quad \Box P \Rightarrow P$
  - 4 $\quad \Box P \Rightarrow \Box\Box P$
  - 5 $\quad \neg\Box P \Rightarrow \Box\neg\Box P$

Possibility Distribution

$$\pi : \Omega \to [0, 1]$$

$\pi(\omega) = 0$ Outright impossible

$\pi(\omega) = 1$ Fully possible, not surprising at all

Normalized: $\exists \omega^* \in \Omega : \pi(\omega^*) = 1$

Remark: this is the fuzzy set of possible states of affairs!

Possibility and Necessity Measures

$$
\begin{aligned}
\Pi(A) &= \max_{\omega \in A} \pi(\omega); \\
N(A) &= 1 - \Pi(\overline{A}) = \min_{\omega \in \overline{A}} \{1 - \pi(\omega)\}.
\end{aligned}
$$

$$
\begin{aligned}
\Pi(\phi) &= \max_{\omega \models \phi} \pi(\omega); \\
N(\phi) &= 1 - \Pi(\neg\phi) = \min_{\omega \not\models \phi} \{1 - \pi(\omega)\}.
\end{aligned}
$$

Properties

Given a **normalized** possibility distribution on a finite universe:

$$\Pi(\bot) = N(\bot) = 0 \qquad \Pi(\top) = N(\top) = 1$$

$$\Pi(\phi \vee \psi) = \max\{\Pi(\phi), \Pi(\psi)\}$$

$$N(\phi \wedge \psi) = \min\{N(\phi), N(\psi)\}$$

$$\Pi(\phi) = 1 - N(\neg\phi) \qquad N(\phi) = 1 - \Pi(\neg\phi)$$

$$N(\phi) \leq \Pi(\phi)$$

$$\Pi(\phi) < 1 \Rightarrow N(\phi) = 0 \qquad N(\phi) > 0 \Rightarrow \Pi(\phi) = 1$$

Possibilistic Logic

$$(\phi, \alpha) \longrightarrow N(\phi) \geq \alpha$$

$$\alpha \in (0, 1]$$

Inference Rules

$(\phi, \alpha) \vdash (\phi, \beta)$ if $\beta \leq \alpha$ (Certainty Weakening)

$(\phi \Rightarrow \psi, \alpha), (\phi, \alpha) \vdash (\psi, \alpha)$ (Modus Ponens)

Weakest-Link Resolution

$$(\neg\phi \vee \psi, \alpha), (\phi \vee \xi, \beta) \vdash (\psi \vee \xi, \min\{\alpha, \beta\})$$

Semantics

$$\Sigma = \{(\phi_i, a_i)\}_{i=1,\ldots,n}$$

$$\pi_\Sigma(\omega) = 1 - \max\{a_i : (\phi_i, a_i) \in \Sigma \text{ and } \omega \models \neg\phi_i\}$$

Probability-Possibility Transformations

Let $p_1 \geq p_2 \ldots \geq p_n$ $\sum_{i=1}^{n} p_i = 1$ be a probability distribution

We can construct a corresponding possibility distribution [Dubois & Prade 1982]:

$$\pi_i = \begin{cases} 1, & \text{if } i = 1; \\ \pi_{i-1}, & \text{if } i > 1 \text{ and } p_i = p_{i-1}; \\ \sum_{j=i}^{n} p_j, & \text{otherwise.} \end{cases}$$

This transformation is reversible, produces a normalized possibility distribution, is **consistent**, preserves the preferences, and is maximally specific.

$$\forall A \subseteq \Omega, \Pi(A) \geq \Pr(A)$$