Logic for AI

Master 1 IFI

Andrea G. B. Tettamanzi
Nice Sophia Antipolis University
Computer Science Department
andrea.tettamanzi@unice.fr

Session 3

Natural Deduction

Agenda

- Non-Compactness and Incompleteness of Herbrand Logic
- Natural Deduction
- The Fitch System

Non-Compactness

Theorem: Herbrand Logic is not compact

Proof:

- Consider the following infinite set of sentences: P(a), P(f(a)), P(f(f(a))), ...
- Assume the vocabulary is {P, a, f}. Hence, the ground terms are a, f(a), f(f(a)), ...
- This set of sentences entails $\forall x\ P(x)$.
- Add in the sentence $\exists x\ \neg P(x)$.
- Clearly, this infinite set is unsatisfiable.
- However, every finite subset is satisfiable.
- Thus, compactness does not hold.

Infinite Proofs

Corollary: In Herbrand Logic, some entailed sentences have only infinite proofs.

Proof.

- The above proof demonstrates a set of sentences that entail $\forall x.p(x)$.
- The set of premises in any finite proof will be missing one of the above sentences.
- Thus, those premises do not entail $\forall x.p(x)$.
- Therefore, no finite proof can exist for $\forall x.p(x)$.

Natural Deduction

- A kind of proof calculus in which logical reasoning is expressed by inference rules closely related to the "natural" way of reasoning.
- This contrasts with Hilbert-style systems, which instead use axioms as much as possible to express the logical laws of deductive reasoning.
- In natural deduction, a proposition is deduced from a collection of premises by repeatedly applying inference rules.
- Gerhard Gentzen and Dag Prawitz laid its foundations
- Fitch notation is a popular notational system for constructing formal proofs in natural deduction

Rule of Inference

- A *schema* is an expression satisfying the grammatical rules of our language except for the occurrence of metavariables (written here as Greek letters) in place of various subparts of the expression.
- Example:

$$\phi \Rightarrow \psi$$

- A rule of inference:

$$\frac{\text{Premises}}{\text{Conclusions}}$$

Linear and Structured Proofs

- A linear proof of a conclusion from a set of premises is a sequence of sentences terminating in the conclusion in which each item is either
  1) a premise
  2) an instance of an axiom schema, or
  3) the result of applying a rule of inference to earlier items in sequence
- Structured proofs differ from linear proofs in that sentences can be grouped into subproofs nested within outer superproofs
  - we can make assumptions within subproofs
  - we can prove conclusions from those assumptions
  - from those derivations, we derive implications in superproofs

Fitch

- Fitch is a proof system that is particularly popular in the Logic community.
- It is as powerful as many other proof systems and is far simpler to use.
- Fitch achieves this simplicity through its support for structured proofs and its use of structured rules of inference in addition to ordinary rules of inference.
- Fitch has fifteen rules of inference in all.
  - Nine of these are ordinary rules of inference.
  - One rule (Implication Introduction) is a structured rule of inference.
  - Five more rules deal with quantifiers

And Introduction and Elimination

And Introduction

$$\dfrac{\begin{array}{l}\phi_1 \\ \vdots \\ \phi_n\end{array}}{\phi_1 \wedge \ldots \wedge \phi_n}$$

And Elimination

$$\dfrac{\phi_1 \wedge \ldots \wedge \phi_n}{\phi_i}$$

Or Introduction and Elimination

Or Introduction

$$\frac{\phi_i}{\phi_1 \vee \ldots \vee \phi_n}$$

Or Elimination

$$\frac{\begin{array}{l} \phi_1 \vee \ldots \vee \phi_n \\ \phi_1 \Rightarrow \psi \\ \vdots \\ \phi_n \Rightarrow \psi \end{array}}{\psi}$$

Negation Introduction and Elimination

Negation Introduction

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi \Rightarrow \neg\psi\end{array}}{\neg\phi}$$

Negation Elimination

$$\frac{\neg\neg\phi}{\phi}$$

Implication Introduction and Elimination

Implication Introduction

$$\frac{\phi \vdash \psi}{\phi \Rightarrow \psi}$$

subproof

Implication Elimination

$$\frac{\begin{array}{l}\phi \Rightarrow \psi \\ \phi\end{array}}{\psi}$$

Biconditional Introduction and Elimination

Biconditional Introduction

$$\frac{\phi \Rightarrow \psi \qquad \psi \Rightarrow \phi}{\phi \Leftrightarrow \psi}$$

Biconditional Elimination

$$\frac{\phi \Leftrightarrow \psi}{\phi \Rightarrow \psi \qquad \psi \Rightarrow \phi}$$

Rules for Universal Quantifier

Universal Introduction

$$\frac{\phi}{\forall \nu . \phi}$$

Where *v* does not occur free in both ϕ and an active assumption

Universal Elimination

$$\frac{\forall \nu . \phi[\nu]}{\phi[\tau]}$$

Rules for Existential Quantifier

Existential Introduction

$$\frac{\phi[\tau]}{\exists\nu.\phi[\nu]}$$

Existential Elimination

$$\frac{\exists\nu.\phi[\nu_1,\ldots,\nu_n,\nu]}{\phi[sk(\nu_1,\ldots,\nu_n)]}$$

(special case)

$$\frac{\exists\nu.\phi[\nu]}{\phi[\tau']}$$

Domain Closure

For languages with finite Herbrand base

$$\frac{\begin{array}{l}\phi[\sigma_1] \\ \vdots \\ \phi[\sigma_n]\end{array}}{\forall \nu . \phi[\nu]}$$

For languages with infinite Herbrand base, we need induction!

Constructing Proofs with the Fitch System

- Constructing proofs using the Fitch system can often be hard and unintuitive, especially for those who encounter it for the first time
- Here are a few guidelines/strategies one can follow
- Based on the properties
  - of the Goal (what is to be proved, the thesis)
  - of the Premises (the assumptions, the hypothesis)

Guidelines Based on the Goal

- Goal is of the form $\varphi \Rightarrow \psi$
  - Assume φ
  - Prove ψ
  - Apply Implication Introduction to prove $\varphi \Rightarrow \psi$
- Goal is of the form $\neg\varphi$
  - Assume φ (*per absurdum*)
  - Find a sentence ψ s.t. you can prove $\varphi \Rightarrow \psi$ and $\varphi \Rightarrow \neg\psi$
  - Apply Negation Introduction to prove $\neg\varphi$
- Goal is of the form φ (with no negation on the outside)
  - Assume $\neg\varphi$ and proceed in a similar manner to prove $\neg\neg\varphi$
  - Apply Negation Elimination on the result $\neg\neg\varphi$ to prove φ

Guidelines Based on the Goal

- Goal is of the form $\varphi_1 \vee \varphi_2 \dots \vee \varphi_n$
  - Prove any φ_i $(1 \leq i \leq n)$
  - Apply OR Introduction to prove $\varphi_1 \vee \varphi_2 \dots \vee \varphi_n$
- Goal is of the form $\varphi_1 \wedge \varphi_2 \dots \wedge \varphi_n$
  - Prove φ_i for every i, $1 \leq i \leq n$
  - Apply AND Introduction to prove $\varphi_1 \wedge \varphi_2 \dots \wedge \varphi_n$

Guidelines Based on the Premises

- There exists a Premise of the form $\varphi \Rightarrow \psi$ and the Goal is ψ
  - Prove φ
  - Apply Implication Elimination on φ and $\varphi \Rightarrow \psi$ to prove ψ

- There exists a Premise of the form $\varphi_1 \vee \varphi_2 \ldots \vee \varphi_n$ and the Goal is ψ
  - Prove $\varphi_i \Rightarrow \psi$ for every i, $1 \leq i \leq n$
  - Apply OR Elimination to prove $\varphi_1 \vee \varphi_2 \ldots \vee \varphi_n \Rightarrow \psi$
  - Apply Implication Elimination on the above result and the premise to prove ψ

Thank you for your attention