Logic for AI

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Session 5

Fuzzy Logic

Agenda

- Introduction
- Fuzzy Set Theory
- Extension Principle
- Fuzzy Logic

Introduction

- Fuzzy Logic:
  - In the broad sense: a mathematical theory to treat imprecision and the vague notions of natural language
  - In the narrow sense: a many-valued logic based on this theory
- Introduced by Lotfi A. Zadeh in 1965
- Basic idea: replace the two truth values T and F with a continuous truth degree taking values between 0 (outright false) et 1 (fully true)
- Fuzzy set theory
  - The extension of classical Logic is based on the definition of a set

Fuzzy Sets

- A “classic” or “crisp” set is completely specified by a characteristic function $\chi : U \rightarrow \{0, 1\}$, such that, for all $x \in U$,
  - $\chi(x) = 1$, if and only if x belongs to the set
  - $\chi(x) = 0$, otherwise.
- To define a “fuzzy” set, we replace χ by a membershi function $\mu : U \rightarrow [0, 1]$, such that, for all $x \in U$,
  - $0 \leq \mu(x) \leq 1$ is the degree to which x belongs to the set
- Since function μ completely specifies the set, we can say that μ **is** the set
- A crisp set is a special case of a fuzzy set!
- The set U is the universe of set μ

Representation

Finite universe:

$$A = \sum_{x \in U} \frac{\alpha_x}{x}$$

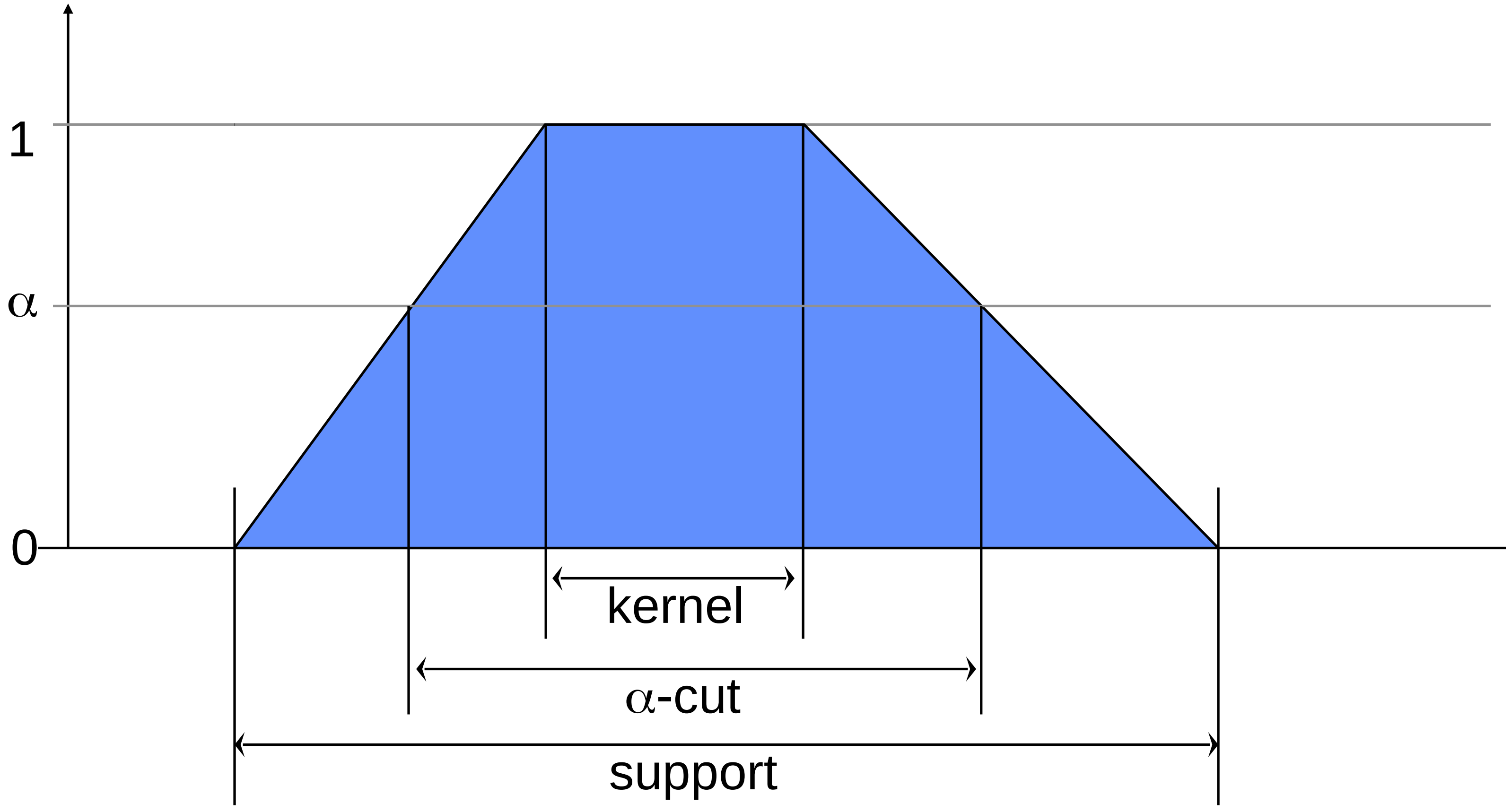
SportsCarBrand = 0.8/BMW + 1/Ferrari + 0/Fiat + 0.5/Mercedes + …

Infinite universe:

$$A = \int_{x \in U} \frac{\mu(x)}{x}$$

$$\text{Hot} = \int_{t=-273,15}^{+\infty} \frac{1/(1+e^{\lambda(20-t)})}{t}$$

Fuzzy Sets



Operations on Fuzzy Sets

- Extension of the operations on crisp sets
- Triangular Norms and Co-Norms
- Min and max are a popular choice

$$\begin{aligned}
(A \cup B)(x) &= \max\{A(x), B(x)\} \\
(A \cap B)(x) &= \min\{A(x), B(x)\} \\
\bar{A}(x) &= 1 - A(x)
\end{aligned}$$

Extension Principle

- Let U and V be two universes and $f : U \rightarrow V$ a mapping
- Let A be a fuzzy set in U
- We can then define a fuzzy set $B = f(A)$ such that, for all $y \in V$:
  - $B(y) = \max\{A(x) : x \in U, f(x) = y\}$
  - $B(y) = 0$, if y does not belong in the image of f

- This principle makes it possible to “fuzzify” (= define a fuzzy extension) of “crisp” theories
- Example: fuzzy numbers → fuzzy arithmetic

Fuzzy Logic (in the narrow sense)

- Fuzzy set theory allows us to introduce fuzzy propositions and predicates
- For propositions, it suffices to reason in terms of interpretations:
  - An interpretation is defined as the set of propositions that are true
  - Fuzzy proposition: partial truth
- For predicates, we consider the identity between a predicate and the set of logical constants (or terms) that satisfy it (its extension)
  - A fuzzy predicate will thus be a predicate whose extensions is a fuzzy set
- Logical connectives are defined accordingly

Logical Operators

- Let τ be the function assigning to a proposition its truth value
- Let P, Q, and R be propositions
  - $\tau(P \wedge Q) = \min \{\tau(P), \tau(Q)\}$
  - $\tau(P \vee Q) = \max \{\tau(P), \tau(Q)\}$
  - $\tau(\neg P) = 1 - \tau(P)$
- Implication has no univocous definition:
  - $\tau(P \rightarrow Q) = \max\{1 - \tau(P), \tau(Q)\}$, since $P \rightarrow Q = \neg P \vee Q$ [K.-D.]
  - $\tau(P \rightarrow Q) = \min\{\tau(P), \tau(Q)\}$ [Mamdani]
  - $\tau(P) \leq \tau(Q)$ [Zadeh]
  - Etc.

Fuzzy-Rule-BAsed System

- Linguistic Variables and Values

- Fuzzy Clause:

  X is *A*

- Fuzzy Rule:

  IF *antecedent* THEN *consequent*

- Defuzzification methods

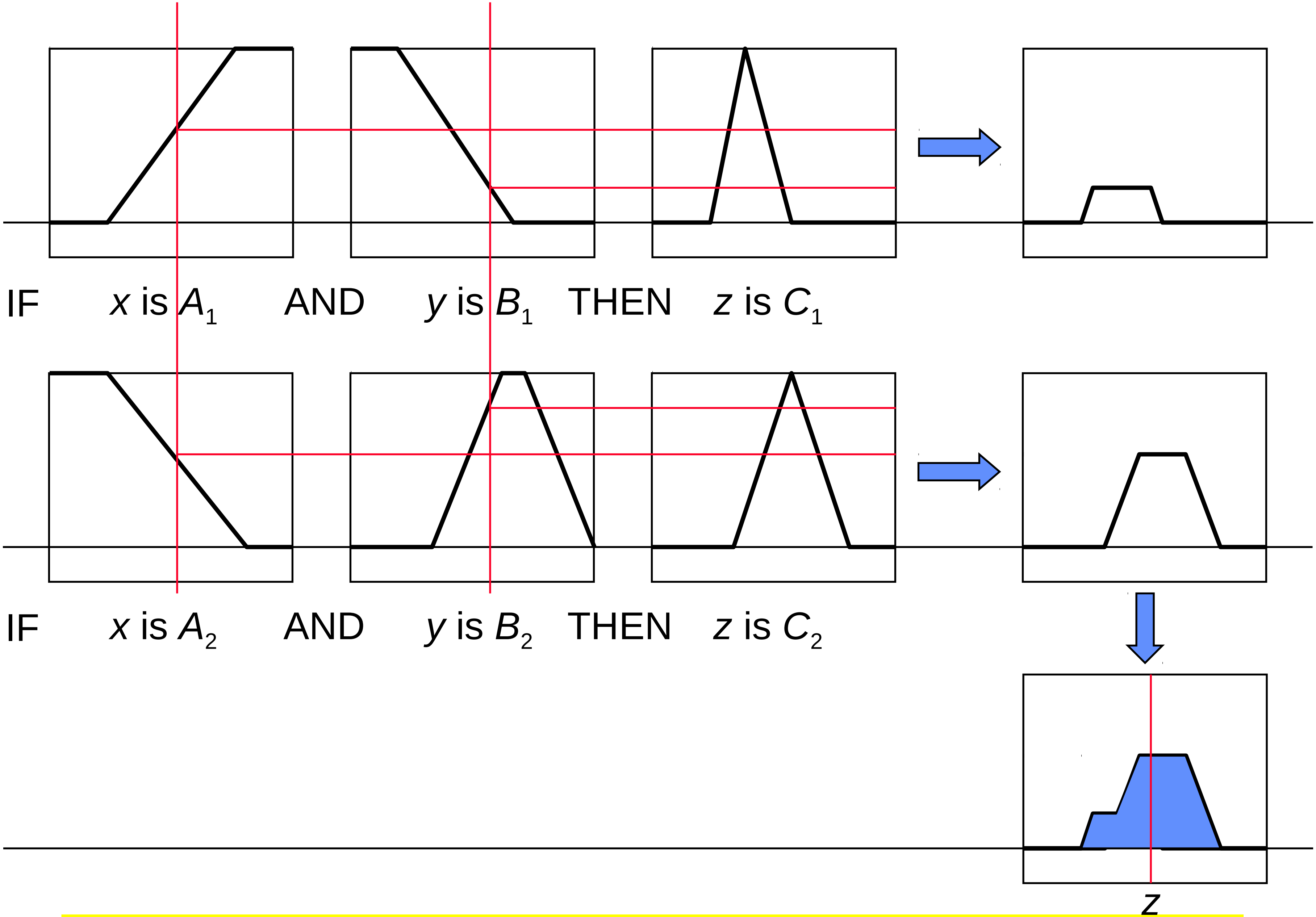
Inference in Fuzzy Rule-Based Systems

Given a set of fuzzy rules

$$
\begin{array}{ll}
\text{IF } P_1(x_1, \ldots, x_n) & \text{THEN } Q_1(y_1, \ldots, y_m), \\
\vdots & \vdots \\
\text{IF } P_r(x_1, \ldots, x_n) & \text{THEN } Q_r(y_1, \ldots, y_m),
\end{array}
$$

The fuzzy set of the values of the dependent variables is given by:

$$
\begin{aligned}
&\tau_R(y_1, \ldots, y_m; x_1, \ldots, x_n) \\
&\quad = \sup_{1 \leq i \leq r} \min\{\tau_{Q_i}(y_1, \ldots, y_m), \tau_{P_i}(x_1, \ldots, x_n)\}.
\end{aligned}
$$


IF x is A1 AND y is B1 THEN z is C1
IF x is A2 AND y is B2 THEN z is C2
z

Fuzzy Set Theory and Probability

- Degrees of membership and probabilities both defined in [0, 1].
- Very similar algebra (e.g., lattice, De Morgan Laws).
- However, they represent two distinct and independent notions:
  - Membership degrees: **imprecision**.
  - Probability: **uncertainty**.

Fuzzy Sets and Probabilities

- The key to understand the difference is the notion of event:
  - A set of elementary events (points in a measurable space);
  - Given an event A:
    - Probability = integral on A of a probability measure;
    - Membership degree = degree to which the result of an experiment or a member of a sample “is” A.

Example (Bezdek 1993)

95%
probability
of being
healthful
and good

95%
membership
in the set of
healthful
and good
drinks

Thank you for your attention