UNIVERSITÉ CÔTE D'AZUR

Logic for AI

Master 1 Informatique

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Unit 3

Predicate Logic: Syntax and Semantics

Agenda

- From Propositional to Predicate Logic
- Relational Logic (= Predicate Logic without Functions)
  - Syntax
  - (Herbrand) Semantics

From Propositional to Predicate Logic

- Propositional Logic
  - Premises:
    - If Jack knows Jill, then Jill knows Jack.
    - Jack knows Jill.
  - Conclusion:
    - Is it the case that Jill knows Jack?
- What about
  - Premises:
    - If one person knows another, then the second person knows the first.
    - Jack knows Jill
  - Conclusion:

Relational Logic

- We need to introduce new features
  - Variables
  - Quantifiers

- Sample sentence

$$\forall x \forall y (\mathsf{knows}(x, y) \Rightarrow \mathsf{knows}(y, x))$$

Relational Logic: Syntax

- Object constants (individuals): Joe, Nice, France, 0, 2345, 3.1415
- Relation constants (predicates): knows, loves, same
- Predicates have an arity:
  - Unary – 1 argument
  - Binary – 2 arguments
  - Ternary – 3 arguments
  - *n*-ary – *n* arguments
- Signature:
  - Set of object constants
  - Set of predicates together with a specification of their arity
- Variables: *x*, *y*, *z*, etc.

Terms and Sentences

- A term is a either a variable or an object constant
- Terms represent objects
- Terms are analogous to noun phrases in natural language
- Sentences
  - Relational sentences (atoms, ≈ simple propositions):
    - A predicate of arity *n* applied to *n* terms
  - Logical sentences (≈ complex propositions):
    - Combinations of sentences using logical operators
  - Quantified sentences:
    - Sentences with quantified variables

Definition (Sentence)

Sentence ::=

- A relation constant with arity *n* applied to *n* terms.
- (¬ φ) where φ is a sentence.
- (φ ∨ ψ), where φ and ψ are sentences.
- (φ ∧ ψ), where φ and ψ are sentences.
- (φ ⇒ ψ), where φ and ψ are sentences.
- (φ ⇔ ψ), where φ and ψ are sentences.
- (∀x φ), where φ is a sentence.
- (∃x φ), where φ is a sentence.

Only expressions produced by the above rules are sentences.

Some Nomenclature

- An **atom** is a relation constant with arity *n* applied to *n* terms
- A **literal** is either an atom or the negation of an atom
- A **ground** sentence has no variables or quantifiers
- A variable is **bound** if and only if it lies within the scope of a quantifier of that variable
- A variable is **free** if it is not bound
- A **closed** sentence has no free variables
- An **open** sentence does have free variables
- We treat free variables in an open sentence as being implicitly universally quantified.

Relational Logic: Semantics

- The classical semantics of relational logic is based on seminal work by Polish logician Alfred Tarski in the 1930s
- Tarski, "Pojęcie prawdy w językach nauk dedukcyjnych" 1933, → "Der Wahrheitsbegriff in den formalisierten Sprachen", 1935
- We will refer to his approach as “Tarskian semantics”

- An alternative approach stems from the work by French mathematician Jacques Herbrand (who died at age 23 while mountain-climbing on the Alps)
- We will refer to this approach as “Herbrand semantics”

Herbrand Base

- The **Herbrand base** for a Relational language is the set of all ground relational sentences that can be formed from the vocabulary of the language
- Example:
  - Object constants: a, b
  - Unary predicate: P
  - Binary predicate: R
  - Herbrand base: { P(a), P(b), R(a,a), R(a,b), R(b,a), R(b,b) }

Interpretation

- An interpretation is a mapping from the Herbrand base (i.e., the ground atoms) to the truth values {F, T}.
- We will use 1 as a synonym for T and 0 as a synonym for F.

$$\mathcal{I} : H \to \{0, 1\}$$

Example: let

$$H = \{P(a), P(b), R(a,a), R(a,b), R(b,a), R(b,b)\}$$

$$P(a)^{\mathcal{I}} = 1 \quad P(b)^{\mathcal{I}} = 0 \quad R(a,a)^{\mathcal{I}} = 1$$
$$R(a,b)^{\mathcal{I}} = 0 \quad R(b,a)^{\mathcal{I}} = 1 \quad R(b,b)^{\mathcal{I}} = 0$$

Equivalent view: interpretation as a subset of the Herbrand base

Sentential Interpretation

- A sentential interpretation is an extension of an interpretation mapping every sentence to the truth values 0 or 1.

$$
\begin{aligned}
P(a)^{\mathcal{I}} &= 1 \\
P(b)^{\mathcal{I}} &= 0
\end{aligned}
\qquad
\begin{aligned}
(P(a) \lor P(b))^{\mathcal{I}} &= 1 \\
(\neg P(b) \land P(a))^{\mathcal{I}} &= 1
\end{aligned}
$$

- Each interpretation is extended to a sentential interpretation based on the type of sentence.

Logical Sentences

$(\neg\phi)^{\mathcal{I}} = 1$	if and only if	$\phi^{\mathcal{I}} = 0$
$(\phi \wedge \psi)^{\mathcal{I}} = 1$	if and only if	$\phi^{\mathcal{I}} = 1$ and $\psi^{\mathcal{I}} = 1$
$(\phi \vee \psi)^{\mathcal{I}} = 1$	if and only if	$\phi^{\mathcal{I}} = 1$ or $\psi^{\mathcal{I}} = 1$
$(\phi \Rightarrow \psi)^{\mathcal{I}} = 1$	if and only if	$\phi^{\mathcal{I}} = 0$ or $\psi^{\mathcal{I}} = 1$
$(\phi \Leftrightarrow \psi)^{\mathcal{I}} = 1$	if and only if	$\phi^{\mathcal{I}} = \psi^{\mathcal{I}}$

Quantified Sentences

- A universally quantified sentence is true under an interpretation if and only if every instance of the scope of the quantified sentence is true under that interpretation.
- An existentially quantified sentence is true under an interpretation if and only if some instance of the scope of the quantified sentence is true under that interpretation.
- An interpretation satisfies a sentence with free variables if and only if it satisfies every instance of that sentence.

Alternative (Equivalent) View

- An interpretation (or model) M is a subset of H
- $\models_M P(t_1,\ldots,t_n)$ if and only if $P(t_1,\ldots,t_n) \in M$.
- $\models_M \neg \psi$ if and only if $\not\models_M \psi$.
- $\models_M \varphi \wedge \psi$ if and only if $\models_M \varphi$ and $\models_M \psi$.
- $\models_M \varphi \vee \psi$ if and only if $\models_M \varphi$ or $\models_M \psi$.
- $\models_M \varphi \Rightarrow \psi$ if and only if $\not\models_M \varphi$ or $\models_M \psi$.
- $\models_M \varphi \Leftarrow \psi$ if and only if $\models_M \psi \Rightarrow \varphi$.
- $\models_M \varphi \Leftrightarrow \psi$ if and only if either $\models_M \varphi \wedge \psi$ or $\models_M \neg\varphi \wedge \neg\psi$.
- $\models_M \forall x.\varphi(x)$ if and only if $\models_M \varphi(t)$ for all ground terms t.
- $\models_M \exists x.\varphi(x)$ if and only if $\models_M \varphi(t)$ for some ground term t.

Quantificational Tautologies

Common Quantifier Reversal:

$$\forall x \forall y P(x,y) \Leftrightarrow \forall y \forall x P(x,y)$$
$$\exists x \exists y P(x,y) \Leftrightarrow \exists y \exists x P(x,y)$$

Existential Distribution:

$$\exists x \forall y P(x,y) \Leftrightarrow \forall y \exists x P(x,y)$$

Negation Distribution:

$$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x)$$

Thank you for your attention