

Logic for AI

Master 1 Informatique

Andrea G. B. Tettamanzi
Laboratoire I3S – Pôle SPARKS
`andrea.tettamanzi@univ-cotedazur.fr`



Unit 8

Fuzzy Logic

Agenda

- Introduction
- Fuzzy Set Theory
- Extension Principle
- Fuzzy Logic

Introduction

- Fuzzy Logic:
 - In the broad sense: a mathematical theory to treat imprecision and the vague notions of natural language
 - In the narrow sense: a many-valued logic based on this theory
- Introduced by Lotfi A. Zadeh in 1965
- Basic idea: replace the two truth values T and F with a continuous truth degree taking values between 0 (outright false) et 1 (fully true)
- Fuzzy set theory
 - The extension of classical Logic is based on the definition of a set

Fuzzy Sets

- A “classic” or “crisp” set is completely specified by a characteristic function $\chi : U \rightarrow \{0, 1\}$, such that, for all $x \in U$,
 - $\chi(x) = 1$, if and only if x belongs to the set
 - $\chi(x) = 0$, otherwise.
- To define a “fuzzy” set, we replace χ by a membership function $\mu : U \rightarrow [0, 1]$, such that, for all $x \in U$,
 - $0 \leq \mu(x) \leq 1$ is the degree to which x belongs to the set
- Since function μ completely specifies the set, we can say that μ **is** the set
- A crisp set is a special case of a fuzzy set!
- The set U is the universe of set μ

Representation

Finite universe:

$$A = \sum_{x \in U} \frac{\alpha_x}{x}$$

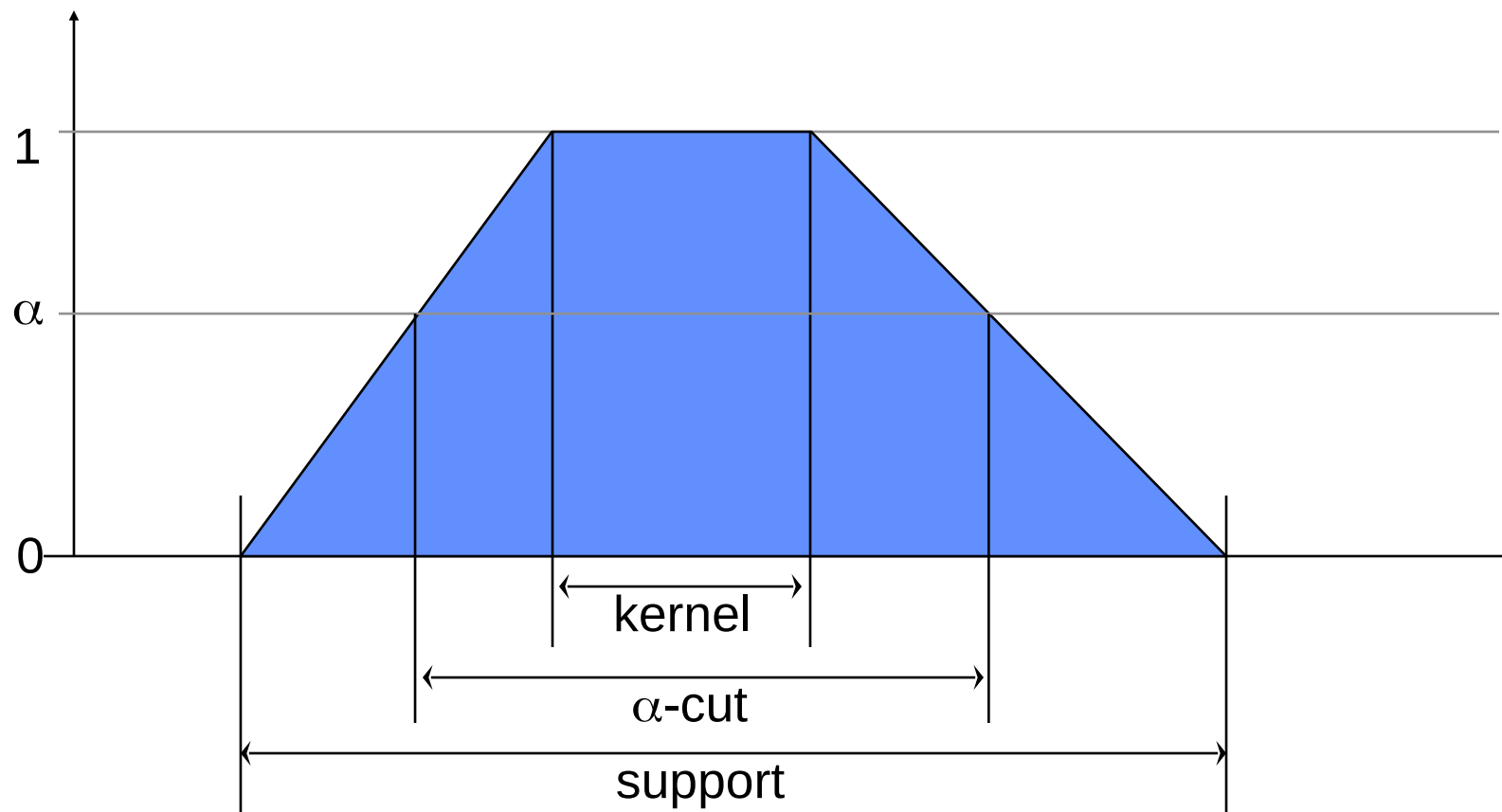
SportsCarBrand = 0.8/BMW + 1/Ferrari + 0/Fiat + 0.5/Mercedes + ...

Infinite universe:

$$A = \int_{x \in U} \frac{\mu(x)}{x}$$

$$\text{Hot} = \int_{t=-273,15}^{+\infty} \frac{1/(1 + e^{\lambda(20-t)})}{t}$$

Fuzzy Sets



Operations on Fuzzy Sets

- Extension of the operations on crisp sets
- Triangular Norms and Co-Norms
- Min and max are a popular choice

$$\begin{aligned}(A \cup B)(x) &= \max\{A(x), B(x)\} \\ (A \cap B)(x) &= \min\{A(x), B(x)\} \\ \bar{A}(x) &= 1 - A(x)\end{aligned}$$

Fuzzy Relations

- An n -ary relation is a subset of the Cartesian product of n sets

$$R(X_1, X_2, \dots, X_n) \subseteq X_1 \times X_2 \times \dots \times X_n$$

- Likewise, a fuzzy relation is a fuzzy subset of the Cartesian product of sets

$$R(x_1, x_2, \dots, x_n) \in [0, 1]$$

- Example: let $X = \{\text{NYC}, \text{London}, \text{Paris}\}$ and let's define binary relation “very far” on X^2

Fuzzy Cartesian Product

- Let $A_1, A_2, \dots, A_n$ be fuzzy sets in $X_1, X_2, \dots, X_n$, respectively
- The fuzzy Cartesian product of $A_1, A_2, \dots, A_n$ is a fuzzy relation on the product space, defined by its membership function

$$(A_1 \times A_2 \times \dots \times A_n)(x_1, x_2, \dots, x_n) = \min\{A_1(x_1), A_2(x_2), \dots, A_n(x_n)\}$$

- Of course, the “min” operator can be replaced by any triangular norm

Extension Principle

- Let U and V be two universes and $f : U \rightarrow V$ a mapping
- Let A be a fuzzy set in U
- We can then define a fuzzy set $B = f(A)$ such that, for all $y \in V$:
 - $B(y) = \max\{A(x) : x \in U, f(x) = y\}$
 - $B(y) = 0$, if y does not belong in the image of f
- This principle makes it possible to “fuzzify” (= define a fuzzy extension) of “crisp” theories
- Example: fuzzy numbers $\rightarrow$ **fuzzy arithmetic**

Fuzzy Logic (in the narrow sense)

- Fuzzy set theory allows us to introduce fuzzy propositions and predicates
- For propositions, it suffices to reason in terms of interpretations:
 - An interpretation is defined as the set of propositions that are true
 - Fuzzy proposition: partial truth
- For predicates, we consider the identity between a predicate and the set of logical constants (or terms) that satisfy it (its extension)
 - A fuzzy predicate will thus be a predicate whose extensions is a fuzzy set
- Logical connectives are defined accordingly

Logical Operators

- Let τ be the function assigning to a proposition its truth value
- Let P , Q , and R be propositions
 - $\tau(P \wedge Q) = \min \{\tau(P), \tau(Q)\}$
 - $\tau(P \vee Q) = \max \{\tau(P), \tau(Q)\}$
 - $\tau(\neg P) = 1 - \tau(P)$
- Implication has no univocous definition:
 - $\tau(P \rightarrow Q) = \max\{1 - \tau(P), \tau(Q)\}$, since $P \rightarrow Q = \neg P \vee Q$ [K.-D.]
 - $\tau(P \rightarrow Q) = \min\{\tau(P), \tau(Q)\}$ [Mamdani]
 - $\tau(P) \leq \tau(Q)$ [Zadeh]
 - Etc.

Fuzzy-Rule-BAsed System

- Linguistic Variables and Values
- Fuzzy Clause:
X is A
- Fuzzy Rule:
IF antecedent THEN consequent
- Defuzzification methods

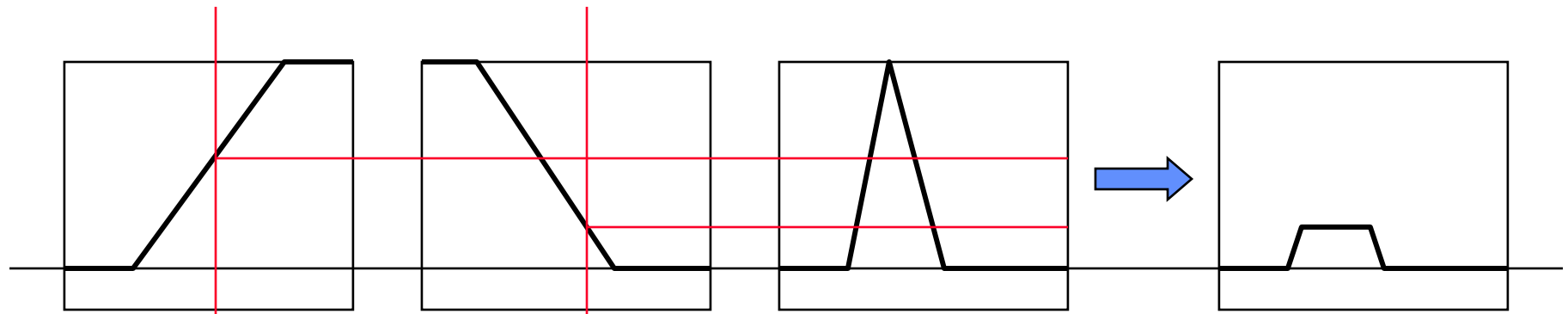
Inference in Fuzzy Rule-Based Systems

Given a set of fuzzy rules

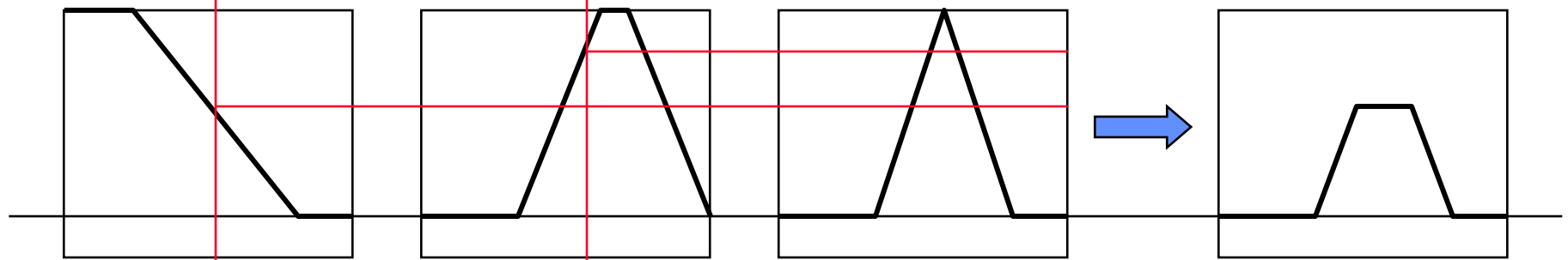
$$\begin{array}{ll} \text{IF } P_1(x_1, \dots, x_n) & \text{THEN } Q_1(y_1, \dots, y_m), \\ \vdots & \vdots \\ \text{IF } P_r(x_1, \dots, x_n) & \text{THEN } Q_r(y_1, \dots, y_m), \end{array}$$

The fuzzy set of the values of the dependent variables is given by:

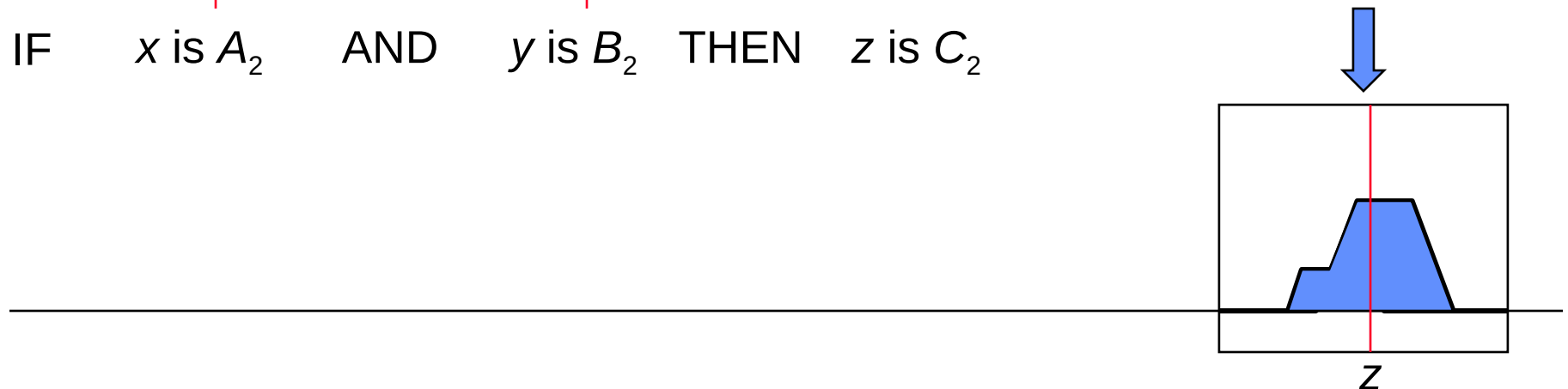
$$\begin{aligned} \tau_R(y_1, \dots, y_m; x_1, \dots, x_n) \\ = \sup_{1 \leq i \leq r} \min\{\tau_{Q_i}(y_1, \dots, y_m), \tau_{P_i}(x_1, \dots, x_n)\}. \end{aligned}$$



IF x is A_1 AND y is B_1 THEN z is C_1



IF x is A_2 AND y is B_2 THEN z is C_2



Fuzzy Set Theory and Probability

- Degrees of membership and probabilities both defined in $[0, 1]$.
- Very similar algebra (e.g., lattice, De Morgan Laws).
- However, they represent two distinct and independent notions:
 - Membership degrees: **imprecision**.
 - Probability: **uncertainty**.

Fuzzy Sets and Probabilities

- The key to understand the difference is the notion of event:
 - A set of elementary events (points in a measurable space);
 - Given an event A:
 - Probability = integral on A of a probability measure;
 - Membership degree = degree to which the result of an experiment or a member of a sample “is” A.

Example (Bezdek 1993)



95%
probability
of being
healthful
and good

95%
membership
in the set of
healthful
and good
drinks

Thank you for your attention

