CHAPTER 8

Fuzzy Quantification in Fuzzy Description Logics

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Abstract

This chapter introduces reasoning procedures for $\mathcal{ALCQ}_F^+(\mathtt{D})$, a fuzzy description logic with extended qualified quantification [D. Sánchez, A.G.B. Tettamanzi, *Generalizing quantification in fuzzy description logics*, in: Proceedings 8th Dortmund Fuzzy Days, Dortmund, Germany, 2004]. The language allows for the definition of fuzzy quantifiers of the absolute and relative kind by means of piecewise linear functions on $\mathbb{N}$ and $\mathbb{Q} \cap [0, 1]$ respectively. In order to reason about instances, the semantics of quantified expressions is defined by using method *GD* [M. Delgado, D. Sánchez, M. Vila, *Fuzzy cardinality based evaluation of quantified sentences*, Int. J. Approximate Reasoning **23** (2000), 23–66], which is based on recently developed measures of the cardinality of fuzzy sets. The main contribution of this chapter is a procedure to calculate the fuzzy satisfiability of a fuzzy concept, which is a very important reasoning task. The procedure considers several different cases and provides direct solutions for the most frequent types of fuzzy concepts. In order to distinguish between these cases, a novel idea of concept independence is also introduced.

Keywords

fuzzy description logics, fuzzy quantification, fuzzy cardinality, fuzzy satisfiability

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1. Introduction

Description logics (DLs) [1] are well-suited for the representation of and reasoning about terminological knowledge, configurations, ontologies, database schemata, etc. The need of expressing and reasoning with imprecise knowledge and the difficulties arising in classifying individuals with respect to an existing terminology motivates research on non-classical DL semantics, suited for these purposes.

To cope with this problem, fuzzy DLs have been proposed that allow for imprecise concept description by using fuzzy sets and fuzzy relations. For instance, a fuzzy version of the description logic $\mathcal{SHOIN}(\mathbf{D})$ is described in Chapter 4 of this book [11]. Work is under way to provide a sound and complete calculus for it as an extension of the one developed for the fuzzy counterpart of the description logic $\mathcal{ALC}$ [10], with complete algorithms for solving the entailment problem, the subsumption problem, as well as the best truth-value bound problem.

However, little attention so far has been paid to the quantification issue. For instance, the fuzzy counterpart of $\mathcal{SHOIN}(\mathbf{D})$ extends the semantics of $\exists$, $\forall$ and *crisp* number restrictions. However, not only concepts, but also quantifiers can be imprecise (e.g., "around two", "most") in many situations typical of real-world applications.

For example, suppose you are the marketing director of a supermarket chain. You are about to launch a new line of low-calorie products. In order to set up your budget, you need to project the sales of this new line of products. This can be done either by means of an expensive market research, or by means of some kind of inference based on your knowledge of customer habits. For instance, you could expect prospective buyers of this new line of products to be essentially faithful customers who mostly buy foods with low energy value. We have here all the ingredients of imprecise knowledge: a "faithful customer" is a fuzzy concept; "low" energy value is a linguistic value, which might be modeled as a fuzzy number; to "mostly" buy a given kind of product is equivalent to a quantified statement of the form "most of the bought products are of this kind", where "most" is an imprecise quantifier.

In [9] we introduced $\mathcal{ALCQ}_F^+$, a fuzzy description logic with extended qualified quantification[1] that allows for the definition of fuzzy quantifiers of the absolute and relative kind by means of piecewise linear functions on $\mathbb{N}$ and $\mathbb{Q} \cap [0, 1]$ respectively. These quantifiers extends the usual (qualified) $\exists$, $\forall$ and number restrictions.

Zadeh [13] showed that imprecise quantifiers can be defined by using fuzzy sets, and by incorporating them into the language and providing the tools to define their semantics we can provide a very powerful knowledge representation tool, with greater expressive power, and closer to the humans' way of thinking.

[1] In keeping with DL naming conventions, the superscript plus is to suggest that, in addition to qualified number restrictions available in the description logic $\mathcal{ALCQ}$ introduced by De Giacomo and Lenzerini [2], we provide also more general fuzzy linguistic quantifiers. The subscript F means that the language deals with infinitely many truth-values, as in the language $\mathcal{ALC}_{F_M}$ of Tresp and Molitor [12].

2. Fuzzy description logics

The fundamental definitions of fuzzy DLs are given in Chapter 4 of this book [11]. Here, we will just point out and comment on some slight differences arising in a few definitions that are largely matter of subjective preferences.

From a notational point of view, we prefer to use the symbols n, m, etc., to stand for natural numbers and α, β, etc., for degrees of truth; we will reserve upper-case Greek letters like Φ and Ψ for fuzzy assertions.

2.1. *Fuzzy assertions and truth-value bounds*

For us, a *fuzzy assertion* has exactly the same syntax of an assertion in a crisp DL: what makes it a *fuzzy* assertion is the fact that it is an assertion about a fuzzy concept or role, and it can thus have a truth value in $[0, 1]$. A fuzzy assertion Ψ thus defined, has to be distinguished from what Straccia calls a *fuzzy concept or role assertion axiom*, of the form $\langle \Psi \leqslant \alpha \rangle$ or $\langle \Psi \geqslant \alpha \rangle$: these could as well be called *truth-value bounds* (TVBs). One important thing to keep in mind is that a TVB is either true or false in a given interpretation; therefore, whereas it is a proposition *about* a fuzzy assertion, a TVB is itself a *crisp* logical proposition.

2.2. *Fuzzy terminological axioms*

Axioms and queries can be of two kinds: specializations and definitions.

A fuzzy concept specialization is a statement of the form $C \sqsubseteq D$, where C and D are concepts. A fuzzy interpretation $\mathcal{I}$ satisfies a fuzzy concept specialization $C \sqsubseteq D$ if, for all $a \in \Delta^{\mathcal{I}}$, $C^{\mathcal{I}}(a) \leqslant D^{\mathcal{I}}(a)$.

A fuzzy concept definition is a statement of the form $C \equiv D$, which can be understood as an abbreviation of the pair of statements $\{C \sqsubseteq D, D \sqsubseteq C\}$.

If it were not for the fuzzy generalized quantifiers that we are going to introduce, the TBox of a fuzzy knowledge base would have formally nothing fuzzy with it, for the syntax of the terminological part of the fuzzy $\mathcal{ALC}$ language as defined in [10] is identical to its crisp counterpart; only the ABox can contain fuzzy assertions.

Admitting TVBs to be part of the fuzzy TBox, as is the case with $\mathcal{SHOIN}(\mathtt{D})$, is equivalent to allowing the formulation of *uncertain* axioms, i.e. axioms whose truth is not absolute. This is a type of uncertainty in a sense *orthogonal* to the uncertainty modeled by means of fuzzy generalized quantification.

3. Cardinality and fuzzy quantification

Crisp quantification is strongly linked to crisp cardinality since a crisp quantifier Q represents a crisp subset of absolute (values in $\mathbb{N}$) or relative (values in $\mathbb{Q} \cap [0, 1]$) cardinalities, we call $S(Q)$. By the same token, fuzzy quantification is strongly linked to fuzzy cardinality.

We may consider two different kinds of cardinalities: *absolute* cardinality measures the number of elements in a set, while *relative* cardinality measures the percentage of elements of one set that also belong in another set (called *referential*).

3.1. *Absolute cardinality*

The most widely used definition of fuzzy set cardinality introduced in [7] is the following: given a fuzzy set F defined on a countable universe set Δ,

$$|F| = \sum_{x \in \Delta} F(x), \tag{1}$$

which is, in general, a real number, and not an integer as it is the case with classical set cardinality. This definition lends itself to many objections and leads to paradoxes. A more satisfactory cardinality measure is *ED*, introduced in [4].

DEFINITION 3.1 [4]. The fuzzy cardinality of a set G is the set $ED(G)$ defined for each $0 \leqslant k \leqslant |\text{supp}(G)|$ as

$$ED(G)(k) = \begin{cases} \alpha_i - \alpha_{i+1}, & \alpha_i \in \Lambda(G) \text{ and } |G_{\alpha_i}| = k, \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

with G_α being the α-cut of G and $\Lambda(G)$ the level set of G, i.e.

$$\Lambda(G) = \big\{G(x) \mid x \in \Delta\big\} \cup \{1\}$$

assuming $\Lambda(G) = \{\alpha_1, \ldots, \alpha_p\}$ and $\alpha_i > \alpha_{i+1}$ for every $i \in \{1, \ldots, p\}$, and $\alpha_{p+1} = 0$.

3.2. *Relative cardinality*

An extension to the *ED* measure to cope with relative cardinality is *ER* introduced in [4].

DEFINITION 3.2 [4]. The fuzzy relative cardinality of a set G with respect to a set F is the set $ER(G/F)$ defined for each $0 \leqslant q \leqslant 1$ as

$$ER(G/F)(q) = \sum_{\alpha_i | C(G/F, \alpha_i) = q} (\alpha_i - \alpha_{i+1}) \tag{3}$$

with

$$C(G/F, \alpha_i) = \frac{|G \cap F|}{|F|}$$

with $\Lambda(F) \cup \Lambda(G \cap F) = \{\alpha_1, \ldots, \alpha_p\}$ and $\alpha_i > \alpha_{i+1}$ for every $i \in \{1, \ldots, p\}$, and $\alpha_{p+1} = 0$.

ER is an extension of *ED* since, if F is crisp and $|F| = n$, then $ER(G/F)(k/n) = ED(G)(k)$ [4].

3.3. *Fuzzy quantifiers*

The concept of fuzzy linguistic quantifier is due to L.A. Zadeh [13]. Fuzzy quantifiers are linguistic labels representing imprecise quantities or percentages. It is usual to distinguish absolute and relative quantifiers, based respectively on absolute and relative cardinality.

Absolute quantifiers represent imprecise natural quantities and are modeled as fuzzy subsets of the naturals, whilst relative quantifiers represent imprecise percentages and are modeled as fuzzy subsets of the rational interval [0, 1].

3.4. *Evaluation of quantified sentences*

Quantified sentences are natural language sentences involving fuzzy linguistic quantifiers, and therefore they express claims about the (fuzzy) quantity or percentage of elements of a (possibly fuzzy) set that verify a certain imprecise property.

According to Zadeh [13], there are two main types of quantified sentences, whose general structure is the following:

Type I sentences: Q of X are G

Type II sentences: Q of F are G

where Q is a linguistic quantifier, X is a crisp finite set, and F and G are two fuzzy subsets of X that represent imprecise properties. Examples are

Type I: *Around* 30 *students are young*

Type II: *Most of the proficient students are young*

Type I sentences are suitable for both absolute and relative quantifiers, whilst type II sentences only make sense for relative quantifiers (i.e. a sentence like "Around 2 F are G" is in fact a type I sentence "Around 2 X are $F \cap G$").

The evaluation of a quantified sentence is the process of calculating its fuzzy accomplishment degree. There are several methods available in the literature (some methods are discussed in [3], recent developments are [6,5]).

In order to extend fuzzy DLs with absolute and relative quantifiers, we shall employ method *GD* [3]. This method calculates the accomplishment degree of a quantified sentence "Q of F are G" as the compatibility degree between the fuzzy relative cardinality measure $ER(G/F)$ [4] and Q (if Q is relative) or between its absolute counterpart $ED(G \cap F)$ [4] and Q (if Q is absolute). A convenient formulation of *GD* is the following [3]:

DEFINITION 3.3. The method *GD* obtains the evaluation of a quantified sentence "Q of F are G" as

$$GD_Q(G/F) = \sum_{\alpha_i \in \Lambda(G/F)} (\alpha_i - \alpha_{i+1}) Q\left(\frac{|(G \cap F)_{\alpha_i}|}{|F_{\alpha_i}|}\right) \tag{4}$$

for relative quantifiers, and

$$GD_Q(G/F) = \sum_{\alpha_i \in \Lambda(G/F)} (\alpha_i - \alpha_{i+1}) Q\big(\big|(F \cap G)_{\alpha_i}\big|\big) \tag{5}$$

for absolute ones, where $F \cap G$ is computed using the minimum, and $\Lambda(G/F) = \Lambda(G \cap F) \cup \Lambda(F)$. We label these values as $\Lambda(G/F) = \{\alpha_1, \ldots, \alpha_p\}$ with $\alpha_i > \alpha_{i+1}$ for every $i \in \{1, \ldots, p\}$ and $\alpha_{p+1} = 0$.

As discussed in [3], $GD_Q(G/F)$ is undefined for type II sentences when F is not normalized since $F_\alpha = \emptyset$ for at least one $\alpha \in [0, 1]$ (in particular for $\alpha = 1$). Possible solutions to this problem are to normalize F (applying the same factor to $F \cap G$ after) or to define the value of any relative quantifier when the relative cardinality is undefined (we note this value as $Q(u)$, by considering $|\emptyset|/|\emptyset| = u$). In this work we shall employ this last option. In particular, $\exists(u) = 0$ and $\forall(u) = 1$.

4. Extending FDLs with fuzzy quantification

Quantified expressions in DLs are expressions of the form $QR.C$, where Q is a quantifier, R is a role and C is a concept, called sometimes a *qualifier*. These expressions are useful for defining new concepts, like $D \equiv QR.C$, meaning that an individual a is an instance of concept D if Q of the individuals which fill its role R are instances of C.

In crisp DLs, quantification is limited to the classical (relative) quantifiers $\exists$ and $\forall$, as well as to the so-called number restrictions, $\leqslant n$, $\geqslant n$, and $= n$ (crisp absolute quantifiers). In this section we propose a fuzzy description logic capable of representing and reasoning with general absolute and relative quantifiers. The goal is to be able to express such fuzzy definitions as, e.g., $D \equiv QR.C$, where Q is now a fuzzy quantifier, meaning that an individual a is an instance of concept D *to the extent that* Q of the individuals which fill its role R are instances of C.

4.1. *The $\mathcal{ALCQ}_F^+(\mathtt{D})$ language*

The extended language we introduce, $\mathcal{ALCQ}_F^+(\mathtt{D})$, where the symbol $\mathtt{D}$ is to suggest the fact that it uses concrete domains, has the following syntax, where A is an atomic concept, R a role, C a concept, Q_{abs} is an absolute fuzzy quantifier, and Q_{rel} a relative fuzzy quantifier:

$$C \longrightarrow \bot \mid \top \mid A \mid C_1 \sqcap C_2 \mid C_1 \sqcup C_2 \mid \neg C \mid$$
$$\forall R.C \mid \exists R.C \mid Q_{\text{abs}} R.C \mid Q_{\text{rel}} R.C.$$

In this extension, the semantics of fuzzy absolute and relative quantifiers is given by using Straccia's notion of *fuzzy concrete predicate* [11]. For relative quantifiers, a membership degree must be defined for the "undefined" case, which arises when the referential set with respect to which the relative cardinality is calculated is empty. This value could depend on the subjective view of the user or on the application at hand, although it can be conventionally agreed upon for some quantifiers like $\exists$ and $\forall$.

4.2. *Fuzzy knowledge base*

A $\mathcal{ALCQ}_F^+(\mathtt{D})$ fuzzy knowledge base comprises two components: a TBox and an ABox.

A TBox $\mathcal{T}$ consists of a finite set of fuzzy terminological axioms of the form $C \sqsubseteq D$ or $C \equiv D$. We require that the TBox be acyclic, i.e. that all concept symbols on the left-hand side of inclusion axioms appear only once in the TBox.

An ABox $\mathcal{A}$ consists of a finite set of fuzzy assertions of the form $C(a)$ or $R(a, b)$.

4.3. *Fuzzy interpretation*

A fuzzy interpretation $\mathcal{I}$ with respect to a concrete domain $\mathtt{D}$ consists of a non-empty domain $\Delta^{\mathcal{I}}$, disjoint from $\Delta_{\mathtt{D}}$, and a fuzzy interpretation function $\cdot^{\mathcal{I}}$, which maps every atomic concept A onto a fuzzy subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, every atomic role R onto a fuzzy binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and every individual name a onto an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

The mapping $\cdot^{\mathcal{I}}$ is extended to all constructs of the language as follows: for all $a \in \Delta^{\mathcal{I}}$,

$$\begin{aligned}
\bot^{\mathcal{I}}(a) &= 0;\\
\top^{\mathcal{I}}(a) &= 1;\\
(C \sqcap D)^{\mathcal{I}}(a) &= \min\bigl\{C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)\bigr\};\\
(C \sqcup D)^{\mathcal{I}}(a) &= \max\bigl\{C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)\bigr\};\\
(\neg C)^{\mathcal{I}}(a) &= 1 - C^{\mathcal{I}}(a).
\end{aligned}$$

For general quantification, the idea is to directly translate a general expression like $QR.C$ into the usual notation of quantified sentences introduced in the previous section as follows: if Q is an absolute quantifier, the degree to which an individual a verifies concept D is the result of the evaluation of the quantified sentence

$$Q \text{ of } \Delta^{\mathcal{I}} \text{ are } R_a^{\mathcal{I}} \cap C^{\mathcal{I}},$$

whilst for relative quantifiers, it is the result of the evaluation of the sentence

$$Q \text{ of } R_a^{\mathcal{I}} \text{ are } C^{\mathcal{I}},$$

where $R_a^{\mathcal{I}}$ is the projection of fuzzy relation $R^{\mathcal{I}}$ on individual a: for all $b \in \Delta^{\mathcal{I}}$, $R_a^{\mathcal{I}}(b) = R^{\mathcal{I}}(a, b)$.

However, the semantics should be well-defined for infinite domains; this means that

(1) the level set Λ might be infinite;
(2) for relative quantifiers, the referential set might be infinite, making the argument of Q undefined.

The first issue can be settled by expressing the truth value of a quantification by means of an integral over the [0, 1] interval. The second issue requires the use of limits.

As usual, we consider two cases depending on whether the quantifier is absolute:

$$(Q_{\text{abs}} R.C)^{\mathcal{I}}(a) = \int_0^1 Q_{\text{abs}}\bigl(\bigl|\bigl\{b\colon\ R^{\mathcal{I}}(a, b) \geqslant \tau \wedge C^{\mathcal{I}}(b) \geqslant \tau\bigr\}\bigr|\bigr)\,\mathrm{d}\tau, \tag{6}$$

or relative:

$$(Q_{\text{rel}} R.C)^{\mathcal{I}}(a) = \int_0^1 Q_{\text{rel}}\left(\lim_{n\to\infty} \frac{|\{b\colon\ R_n^{\mathcal{I}}(a, b) \geqslant \tau \wedge C^{\mathcal{I}}(b) \geqslant \tau\}|}{|\{b\colon\ R_n^{\mathcal{I}}(a, b) \geqslant \tau\}|}\right)\mathrm{d}\tau, \tag{7}$$

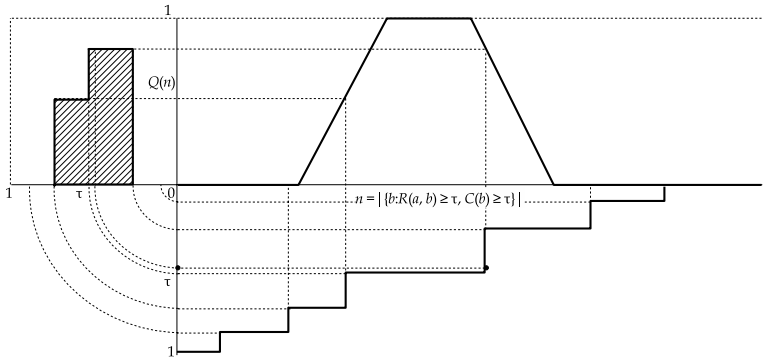

Fig. 1. A graphical depiction of the interpretation of fuzzy concept $QR.C$ with Q an absolute fuzzy quantifier whose membership function is shown in the upper-left part of the figure; the bottom-left part of the figure graphs the cardinality of the τ-cuts of $R^{\mathcal{I}}(a,b) \cap C^{\mathcal{I}}(b)$ for $0 \leqslant \tau \leqslant 1$; the upper-right part of the figure shows the polygon resulting from applying the membership function of Q to the former cardinalities, whose area represents the degree of truth of $QR.C$.

where $\{R_n^{\mathcal{I}}\}_{n=1,2,\ldots}$ is a sequence of fuzzy relations such that

(1) for all $n \geqslant 1$, $R_n^{\mathcal{I}} \subseteq R_{n+1}^{\mathcal{I}}$;
(2) $\lim_{n\to\infty} R_n^{\mathcal{I}} = R^{\mathcal{I}}$;
(3) for all $n \geqslant 1$, $\mathrm{supp}(R_n^{\mathcal{I}})$ is a finite set.

Of course, given any finite interpretation, the above integrals become discrete summations because the arguments of the $Q_{\mathrm{abs}}(\cdot)$ and $Q_{\mathrm{rel}}(\cdot)$ membership functions turn out to be step functions with a finite set of levels, yielding their finite versions

$$(Q_{\mathrm{abs}}R.C)^{\mathcal{I}}(a) = GD_{Q_{\mathrm{abs}}}\left(\left(R_a^{\mathcal{I}} \cap C^{\mathcal{I}}\right)/\Delta^{\mathcal{I}}\right), \tag{8}$$

$$(Q_{\mathrm{rel}}R.C)^{\mathcal{I}}(a) = GD_{Q_{\mathrm{rel}}}\left(C^{\mathcal{I}}/R_a^{\mathcal{I}}\right) \tag{9}$$

where GD is the evaluation method introduced in [3]. Figure 1 depicts an example of how the integral is calculated for the absolute quantifier case.

In particular, the semantics for the quantifiers $\exists$ and $\forall$ is:

$$(\exists R.C)^{\mathcal{I}}(a) = \sum_{(R_a^{\mathcal{I}})_{\alpha_i} \cap (C^{\mathcal{I}})_{\alpha_i} \neq \emptyset} (\alpha_i - \alpha_{i+1}), \tag{10}$$

$$(\forall R.C)^{\mathcal{I}}(a) = \sum_{(R_a^{\mathcal{I}})_{\alpha_i} \subseteq (C^{\mathcal{I}})_{\alpha_i}} (\alpha_i - \alpha_{i+1}) \tag{11}$$

with $\alpha_i \in \Lambda(G/F)$. According to the properties of GD [3], the semantics of the existential quantification turns out to have its usual, intuitive definition, i.e. Equation (10) is equivalent to

$$(\exists R.C)^{\mathcal{I}}(a) = \sup_{b\in\Delta^{\mathcal{I}}} \min\left\{R_a^{\mathcal{I}}(b), C^{\mathcal{I}}(b)\right\}.$$

As a particular case, when the referential F is crisp, GD verifies De Morgan's laws, i.e. $1 - GD_{\forall}(G/F) = GD_{\exists}(\neg G/F)$ and $1 - GD_{\exists}(G/F) = GD_{\forall}(\neg G/F)$. Hence, if $R_a^{\mathcal{I}}$ is crisp then Equation (11) is equivalent to

$$(\forall R.C)^{\mathcal{I}}(a) = \inf_{b\in\Delta^{\mathcal{I}}} \max\left\{1 - R_a^{\mathcal{I}}(b), C^{\mathcal{I}}(b)\right\}.$$

However, this is not true in general: let's consider the fuzzy sets whose membership functions are $R_a^{\mathcal{I}}$ and $C^{\mathcal{I}}$ respectively (we shall use the same notation for the functions and the fuzzy subsets); De Morgan's laws hold only if

$$R_a^{\mathcal{I}} \subseteq C^{\mathcal{I}} \Leftrightarrow R_a^{\mathcal{I}} \cap (\neg C)^{\mathcal{I}} = \emptyset,$$

i.e. only when $R_a^{\mathcal{I}}$ (or $C^{\mathcal{I}}$) is crisp. Otherwise we could have a situation where both sets are fuzzy and $R_a^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ (and hence the evaluation using $\forall$ is expected to yield 1), but $R_a^{\mathcal{I}} \cap (\neg C)^{\mathcal{I}} \neq \emptyset$ (and hence the evaluation using $\exists$ is expected to yield a value greater than 0). For example, suppose that

$$R_a^{\mathcal{I}} = 1/b_1 + 0.6/b_2 + 0.4/b_3,$$
$$C^{\mathcal{I}} = 1/b_1 + 0.9/b_2 + 0.5/b_3.$$

Then $R_a^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ and $R_a^{\mathcal{I}} \cap (\neg C)^{\mathcal{I}} = 0.1/b_2 + 0.4/b_3 \neq \emptyset$, consequently $GD_{\forall}(C^{\mathcal{I}}/R_a^{\mathcal{I}}) = 1$, but $GD_{\exists}((\neg C)^{\mathcal{I}}/R_a^{\mathcal{I}}) = 0.4 \neq 0$.

4.4. *An example*

Let us go back to the example of the marketing director of a supermarket chain about to launch a line of low-calorie products.

The knowledge base describing the business of running a supermarket chain could contain, among others, the following terminological axioms:

$$\mathsf{FaithfulCustomer} \sqsubseteq \mathsf{Customer} \sqsubseteq \top$$
$$\mathsf{FoodProduct} \sqsubseteq \mathsf{Product} \sqsubseteq \top$$
$$\mathsf{LowCalorie} \sqsubseteq \mathsf{EnergyMeasure} \sqsubseteq \top$$
$$\mathsf{LowCalorieFood} \equiv \mathsf{FoodProduct} \sqcap \forall \mathsf{energyValue}.\mathsf{LowCalorie}$$

The ABox describing facts about your supermarket chain might contain TVBs which we might summarize as follows:

- given an individual customer c and a product p, $\mathsf{buys}(c, p)$ might be interpreted as

  $$\mathsf{buys}(c, p) = f\big(\text{weeklyrevenue}(c, p)\big),$$

  where $f : \mathbb{R} \to [0, 1]$ is non-decreasing, and $\text{weeklyrevenue}(c, p) : \mathsf{Customer}^{\mathcal{I}} \times \mathsf{Product}^{\mathcal{I}} \to \mathbb{R}$ returns the result of a database query which calculates the average revenue generated by product p on customer c in all the stores operated by the chain;
- given an individual customer c, $\mathsf{FaithfulCustomer}(c)$ might be interpreted as

  $$\mathsf{FaithfulCustomer}(c) = g\big(\text{weeklyrevenue}(c)\big),$$

  where $g : \mathbb{R} \to [0, 1]$ is non-decreasing, and $\text{weeklyrevenue}(c) : \mathsf{Customer}^{\mathcal{I}} \to \mathbb{R}$ returns the result of a database query which calculates the average revenue generated by customer c in all the stores operated by the chain;

Table 1
The energy value, membership in the LowCalorieFood, and the degree to which customer CARD0400009324198 buys them for a small sample of products.

Product	Energy [kJ/hg]	LowCalorieFood$(\cdot)$	buys(CARD$\ldots,\cdot)$
GTIN8001350010239	1680	0.320	0.510
GTIN8007290330987	1475	0.525	0.050
GTIN8076809518581	1975	0.025	0.572
GTIN8000113004003	1523	0.477	0.210
GTIN8002330006969	498	1.000	1.000
GTIN8005410002110	199	1.000	1.000
GTIN017600081636	1967	0.033	0.184

- finally, LowCalorie(x), where x is an average energy value per 100 g of product measured in kJ, could be interpreted as

$$\mathsf{LowCalorie}(x) = \begin{cases} 1, & x < 1000, \\ \dfrac{2000 - x}{1000}, & 1000 \leqslant x \leqslant 2000, \\ 0, & x > 2000. \end{cases}$$

By using the $\mathcal{ALCQ}_F^+(\mathsf{D})$ language, it is now possible to express the notion of a faithful customer who mostly buys food with low energy value as

$$C \equiv \mathsf{FaithfulCustomer} \sqcap (\mathsf{Most})\mathsf{buys}.\mathsf{LowCalorieFood},$$

where $(\mathsf{Most}) \equiv (0/u + 0/0.5 + 1/0.75)$.

A useful deduction this new axiom allows you to make is, for instance, calculating the extent to which a given individual customer or, more precisely, a fidelity card, say CARD0400009324198, is a C. For instance, you could know that

$$\mathsf{FaithfulCustomer}(\mathsf{CARD0400009324198}) = 0.8,$$

and, by querying the sales database, you might get all the degrees to which that customer buys each product. For sake of example, we give a small subset of those degrees of truth in Table 1, along with the energy values of the relevant products.

According to the semantics of $\mathcal{ALCQ}_F^+(\mathsf{D})$,

$$C(\mathsf{CARD0400009324198}) \approx 0.742,$$

i.e. the degree to which most of the items purchased by this customer are low-calorie is around 0.742. This seems to be in accordance with the data in Table 1, where we can see that four products (those products p in rows 2, 4, 5, and 6) verify

$$\mathsf{buys}(\mathsf{CARD0400009324198}, p) \leqslant \mathsf{LowCalorieFood}(p)$$

while for the products in rows 1 and 7 the difference between being purchased and being low-calorie food is not so high. Only the item in row 3 seems to be a clear case of item purchased but not low-calorie.

As another justification of why this result appears in agreement with the data, in Table 2 we show the percentage of purchased items that are low-calorie at α-cuts of the same level. At any other level, the percentage obtained is one of those shown in Table 2.

Table 2
Percentage of purchased items that are low-calorie at significant levels.

Level	Percentage
1.000	$1.000 = 2/2$
0.572	$0.667 = 2/3$
0.510	$0.500 = 2/4$
0.320	$0.750 = 3/4$
0.210	$0.800 = 4/5$
0.184	$0.667 = 4/6$
0.050	$0.714 = 5/7$
0.033	$0.857 = 6/7$

At many levels the percentage is above 0.75, therefore fitting the concept of Most as we have defined it. At level 0.050 the percentage is almost 0.75. The only level that clearly doesn't fit Most is 0.510, but at the next level (0.320) we have again 0.75 and $\mathsf{Most}(0.75) = 1$.

5. Fuzzy satisfiability

One particularly important reasoning task is to calculate the fuzzy satisfiability of a fuzzy concept D, i.e. the tightest interval of values $\mathcal{S}(D) = [\beta_D, \tau_D]$ such that for any interpretation $\mathcal{I}$ and any object $a \in \mathcal{I}$, the degree of truth of an assertion of the form $D(a)$ under $\mathcal{I}$ verifies $D^{\mathcal{I}}(a) \in [\beta_D, \tau_D]$ (i.e. the maximum interval $[\beta, \tau]$ such that D is $[\beta, \tau]$-satisfiable in the sense of Navara's definition [8]).

One might wonder whether it is really *satisfiability* we are dealing with here, or rather *validity*. The fact is, these two categories, which can be clearly distinguished in the realm of classical logic, become just the two extremes of a continuum in the case of fuzzy logic: as a matter of fact, a fuzzy concept D could be considered *valid* in the classical sense if $\mathcal{S}(D) = \{1\}$, and *satisfiable* (but not valid) in the classical sense if $\mathcal{S}(D) = [0, 1]$. It is all a matter of conventions: one might say, e.g., that, if $\mathcal{S}(D) = [\frac{3}{4}, 1]$, D is more valid than just satisfiable, whereas D might be considered almost unsatisfiable if $\mathcal{S}(D) = [0, \frac{1}{4}]$. Or we might define β_D as the degree of validity of D, and τ_D as the degree of satisfiability of D. However, for the sake of convenience, here we will use the term *fuzzy satisfiability* to cover the whole gamut of such possibilities.

We now illustrate a polynomial-space algorithm to calculate the fuzzy satisfiability of a fuzzy concept. Though infinite interpretations are taken into account in the definition of the semantics of the language $\mathcal{ALCQ}_F^+(\mathsf{D})$, in practice and due to the physical limitations of computers, we are going to deal with a finite number of individuals. The same limitations put a bound on the number of different membership degrees we can deal with. Therefore we shall calculate the fuzzy satisfiability of a fuzzy concept up to a certain precision degree, given as a number of decimals p.

5.1. *Concept independence*

In addition to the general algorithm, we have obtained some results showing that we can calculate the fuzzy satisfiability of a fuzzy concept directly in some (the most common) cases. Some of these results are based on a previous result about *independence* of fuzzy concepts. We introduce the following definition:

DEFINITION 5.1. Two concepts A and B are *independent* of each other if, for all $\alpha \in \mathcal{S}(A)$ and $\beta \in \mathcal{S}(B)$, there exists an interpretation $\mathcal{I}$ containing an individual $d \in \Delta_{\mathcal{I}}$ such that

$$A^{\mathcal{I}}(d) = \alpha \quad \text{and} \quad B^{\mathcal{I}}(d) = \beta.$$

In other words, A and B are independent if the degree of truth of A does not affect the degree of truth of B and *vice versa*.

The following results help to determine whether two concepts are independent.

PROPOSITION 5.1. *Two concepts A and B, not containing quantifiers, which are neither tautologies nor contradictions, are independent if and only if the following four concepts are satisfiable in the crisp sense*: $A \sqcap B$; $A \sqcap \neg B$; $\neg A \sqcap B$; $\neg A \sqcap \neg B$.

PROOF. We first show that independence of A and B implies the satisfiability of the four conjuncts: since A and B are neither tautologies nor contradictions, $\mathcal{S}(A) = \mathcal{S}(B) = [0, 1]$; therefore, it is possible to construct an interpretation with four individuals d_1, d_2, d_3, d_4 such that

$$\begin{aligned} A(d_1) &= 1, \qquad B(d_1) = 1, \\ A(d_2) &= 1, \qquad B(d_2) = 0, \\ A(d_3) &= 0, \qquad B(d_3) = 1, \\ A(d_4) &= 0, \qquad B(d_4) = 0; \end{aligned}$$

now, d_1 satisfies concept 1, d_2 satisfies concept 2, d_3 satisfies concept 3, and d_4 satisfies concept 4: all four conjuncts are satisfiable in the crisp sense.

Next we show that if the four conjuncts are satisfiable in the crisp sense, then A and B are independent. The four conjuncts are satisfiable in the crisp sense if there exists an interpretation $\mathcal{I}$ containing four individuals d_1, d_2, d_3, d_4 such that

$$\begin{aligned} A(d_1) &= 1, \qquad B(d_1) = 1, \\ A(d_2) &= 1, \qquad B(d_2) = 0, \\ A(d_3) &= 0, \qquad B(d_3) = 1, \\ A(d_4) &= 0, \qquad B(d_4) = 0; \end{aligned}$$

now, $\mathcal{S}(A)$ and $\mathcal{S}(B)$ are convex sets; therefore, if $0 \in \mathcal{S}(A)$ and $1 \in \mathcal{S}(A)$, it must be $\mathcal{S}(A) = [0, 1]$; for the same reason, it must be $\mathcal{S}(B) = [0, 1]$. First of all, neither A nor B can be a tautology or a contradiction; furthermore, when $A(d) = 0$, $B(d)$ can be 0 or 1, whence $B(d) \in [0, 1]$; the same holds when $A(d) = 1$. The possible constraints which could bind the truth values $\alpha = A(d)$ and $\beta = B(d)$, for all d, are only three:

- $\alpha \leqslant \beta$, excluded by individual d_2;
- $\beta \leqslant \alpha$, excluded by individual d_3;
- $\alpha = 1 - \beta$, excluded by individuals d_1 and d_4.

We must therefore conclude that for all $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, it is possible to construct an interpretation containing an individual d such that $A(d) = \alpha$ and $B(d) = \beta$. □

PROPOSITION 5.2. *Let D_1 and D_2 be two independent concepts. Then, no atomic concept and no role appears in the expansion of both.*

PROOF. Trivial. □

PROPOSITION 5.3. *Let D_1 be a concept that doesn't contain quantifiers in its expansion, and let $D_2 \equiv QR.C$. Then, D_1 and D_2 are independent, regardless of whether D_1 and C are independent or not.*

PROOF. The truth of $D_2(a)$ in any interpretation depends on Q, R and C, where Q and R don't appear in D_1, and C applies to individuals $b_1, \ldots, b_n$ with $b_i \neq a$. Hence, $\forall \alpha_1 \in \mathcal{S}(D_1)$ and $\alpha_2 \in \mathcal{S}(D_2)$ we can build an interpretation $\mathcal{I}$ such that $D_1^{\mathcal{I}}(a) = \alpha_1$ and $D_2^{\mathcal{I}}(a) = \alpha_2$. □

PROPOSITION 5.4. *Let $D_1 \equiv Q_1 R_1.C_1$ and let $D_2 \equiv Q_2 R_2.C_2$. If $R_1 \neq R_2$ or C_1 and C_2 are independent, then D_1 and D_2 are independent.*

PROOF. We consider both conditions:

- Let $R_1 \neq R_2$. Then we can build an interpretation $\mathcal{I}$ such that $R_1 \cap R_2 = \emptyset$, i.e. the set of individuals b_i^1 related to a by role R_1 is different from the set of individuals b_j^2 related to a by role R_2. Since only degrees of truth of $C_1(b_i^1)$ and $C_2(b_j^2)$ are relevant in order to calculate $D_1(a)$ and $D_2(a)$, and these are independent, both concepts are independent.
- Let $R_1 = R_2$ and let C_1 and C_2 be independent. Then, though the set of individuals b_i related to a in each case is the same, there is no dependence between the degrees we can assign to $C_1(b_i)$ and $C_2(b_i)$ for every i. Hence, D_1 and D_2 are independent. □

A general procedure to determine whether two concepts are independent can be obtained from Proposition 5.1 using the crisp procedure to check unsatisfiability. However, Propositions 5.2, 5.3, and 5.4 can make things easier in some cases.

5.2. *Calculating fuzzy satisfiability*

We shall consider different cases related to the syntactic rules of our language. For each case we want to provide the minimum number of individuals needed and, when it is possible to derive it directly, the fuzzy satisfiability.

Since we are dealing with a fixed task, we shall denote the minimum number of individuals needed to calculate the fuzzy satisfiability of a fuzzy concept D as n_D. In general, $n_D = \min \mathbb{N}_D$, where $\mathbb{N}_D$ is the set of possible cardinalities of $\Delta_{\mathcal{I}}$ that allow us to calculate the fuzzy satisfiability.

Before one starts the process the TBox must be eliminated, which is always possible in $\mathcal{ALC}\mathcal{Q}_F^+(\mathsf{D})$, since the TBox is acyclic.

5.3. *Atomic concepts*

Let A be an atomic concept, then $\mathcal{S}(A) = [0, 1]$ (if A is fuzzy) and $\mathcal{S}(A) = \{0, 1\}$ (if A is crisp[2]). In both cases, the number of individuals required in an interpretation is in the set $\mathbb{N}_A = \{n \in \mathbb{N}\colon\ n \geqslant 1\}$, hence $n_A = 1$.

5.4. *Negation*

We consider several cases:

(1) Let A be an atomic concept. Then $\mathcal{S}(\neg A) = \mathcal{S}(A)$, $\mathbb{N}_A = \mathbb{N}_{\neg A}$, and $n_{\neg A} = n_A$.
(2) $\mathcal{S}(\neg\neg C) = \mathcal{S}(C)$, $\mathbb{N}_C = \mathbb{N}_{\neg\neg C}$ and $n_{\neg\neg C} = n_C$.
(3) Let $C = C_1 \sqcap \cdots \sqcap C_r$. Then $\neg C = \neg C_1 \sqcup \cdots \sqcup \neg C_r$ and $\mathcal{S}(\neg C)$, $\mathbb{N}_{\neg C}$, and $n_{\neg C}$ are calculated by following the procedure we provide for disjunction.
(4) In the same way, let $C = C_1 \sqcup \cdots \sqcup C_r$. Then $\neg C = \neg C_1 \sqcap \cdots \sqcap \neg C_r$ and $\mathcal{S}(\neg C)$, $\mathbb{N}_{\neg C}$, and $n_{\neg C}$ are calculated by following the procedure we provide for conjunction.
(5) Let $D \equiv QR.C$. Then,

$$\mathcal{S}(\neg D) = \mathcal{S}\big(\neg(QR.C)\big) = \mathcal{S}\big((\neg Q)R.C\big), \tag{12}$$

since, by the properties of method *GD*

$$1 - GD_Q(G/F) = GD_{\neg Q}(G/F).$$

In addition, $\mathbb{N}_{\neg D} = \mathbb{N}_{(\neg Q)R.C}$ and $n_{\neg D} = n_{[(\neg Q)R.C]}$.

5.5. *Absolute quantifiers*

Satisfiability of $D \equiv QR.C$ depends on Q and the satisfiability of C. The set $\mathbb{N}_D$ (and hence the value n_D) depends on Q and $\mathbb{N}_C$.

We assume that $\operatorname{core}(Q) \neq \emptyset$ and $\mathbb{N}\backslash\operatorname{supp}(Q) \neq \emptyset$. This means it is always possible to find $n_0^Q, n_1^Q \in \mathbb{N}$ such that $Q(n_0^Q) = 0$ and $Q(n_1^Q) = 1$ and such that $n < n_0^Q$ implies $Q(n) > 0$ and $n < q_1$ implies $Q(n) < 1$.

PROPOSITION 5.5. *Let $D \equiv QR.C$ and let $\mathcal{S}(C) = [\beta_C, \tau_C]$. Then $\mathcal{S}(D) = \mathcal{S}(QR.C) = [\beta_D, \tau_D]$ with*

$$\beta_D = (1 - \tau_C)Q(0), \tag{13}$$

$$\tau_D = \max\big\{\big(\tau_C + (1 - \tau_C)Q(0)\big), Q(0)\big\}. \tag{14}$$

In addition,

$$\mathbb{N}_{QR.C} = \big\{n + n'\colon\ \big(n \geqslant \max\big(n_0^Q, n_1^Q\big)\big) \wedge (n' \in \mathbb{N}_C)\big\}.$$

[2] A concept C is *crisp* if an axiom of the form $C \sqcap \neg C \sqsubseteq \bot$ is found in the TBox.

PROOF. Let a be an individual. We know that $\mathcal{S}(R_a) = [0, 1]$, hence $\mathcal{S}(R_a \cap C) = [0, \tau_C]$. The extreme values for $D(a)$ can be reached for an interpretation such that the possible cardinalities of the set $R_a \cap C$ match the quantifier with degree 0 or 1, and the simplest such interpretation is one where all the individuals b_i verify $(R_a \cap C)(b_i) = (R_a \cap C)(b_j)$ and such that the number of b's is a certain q with $Q(q) = 0$ or $Q(q) = 1$. We distinguish three types of interpretations:

- We introduce n_0^Q individuals $b_1, \ldots, b_{n_0^Q}$ such that $(R_a \cap C)(b_i) = (R_a \cap C)(b_j) = \alpha$ $\forall i, j$. Then $ER(R_a \cap C) = (1-\alpha)/0 + \alpha/n_0^Q$, $Q(n_0^Q) = 0$, and $D(a) = (1-\alpha)Q(0)$. Since $\alpha \in [0, \tau_C]$ we have $D(a) \in [(1-\tau_C)Q(0), Q(0)]$. In this interpretation there are at least $n_0^Q + n_C$ individuals.
- Let $q_1 = 0$. We define $R_a = \emptyset$ and hence $ER(R_a \cap C) = 1/0$ and $D(a) = 1 = Q(0) = \max\{(\tau_C + (1-\tau_C)Q(0)), Q(0)\}$. Here we need just one individual (a).
- Let $q_1 > 0$. We introduce q_1 individuals $b_1, \ldots, b_{n_1^Q}$ such that $(R_a \cap C)(b_i) = (R_a \cap C)(b_j) = \alpha$ $\forall i, j$. Then $ER(R_a \cap C) = (1-\alpha)/0 + \alpha/q_1$ and $D(a) = \alpha + (1-\alpha)Q(0)$. Since $\alpha \in [0, \tau_C]$ and $D(a)$ is non-decreasing in α, we have $D(a) \in [Q(0), \tau_C + (1-\tau_C)Q(0)]$. In this interpretation there are at least $n_1^Q + n_C$ individuals.

Hence,

$$\beta_D = \min\big\{(1-\tau_C)Q(0), Q(0)\big\} = (1-\tau_C)Q(0),$$
$$\tau_D = \max\big\{\big(\tau_C + (1-\tau_C)Q(0)\big), Q(0)\big\}$$

and we need at least $\max(n_0^Q, n_1^Q) + n_C$ individuals. □

COROLLARY 5.1. *If Q is a quantifier with $Q(0) = 0$, or $\tau_C = 1$, then*

$$\mathcal{S}(QR.C) = [0, \tau_C].$$

5.6. *Relative quantifiers*

Again, satisfiability of $D \equiv QR.C$ depends on Q and the satisfiability of C, and the set $\mathbb{N}_D$ (and hence the value n_D) depends on Q and $\mathbb{N}_C$.

We also assume that $\text{core}(Q) \neq \emptyset$ and $[0, 1] \cap \mathbb{Q} \backslash \text{supp}(Q) \neq \emptyset$. This means it is always possible to find $q_0^Q, q_1^Q \in [0, 1] \cap \mathbb{Q}$ such that $Q(q_0^Q) = 0$ and $Q(q_1^Q) = 1$ and such that $q < q_0^Q$ implies $Q(q) > 0$ and $q < q_1$ implies $Q(q) < 1$.

Since $q_0^Q, q_1^Q \in [0, 1] \cap \mathbb{Q}$ it is always possible to find $n_0^Q, d_0^Q, n_1^Q, d_1^Q \in \mathbb{N}$ such that $q_0^Q = \frac{n_0^Q}{d_0^Q}$ and $q_1^Q = \frac{n_1^Q}{d_1^Q}$.[3]

As in the case of absolute quantifiers, in order to calculate the whole range of truth values that form $\mathcal{S}(D)$ we will consider those interpretations such that the number of degrees of truth is the minimum possible, and verifying that all the important cardinalities $(0, 1, q_0^Q, q_1^Q$ and the undefined case $0/0)$ are considered. Hence, we shall consider interpretations of the following kind:

[3] If $q_0^Q = 0$ (resp. $q_1^Q = 0$) we shall assume $d_0^Q = d_1^Q$ and $n_0^Q = 0$ (resp. $n_1^Q = 0$).

- We consider an individual a in order to evaluate $D(a)$.
- The number of individuals related to a by role R, namely d^Q, will be a multiple of both d_0^Q and d_1^Q, in particular $d^Q = mcm(d_0^Q, d_1^Q)$ is the minimum such value. We shall denote such individuals as $b_1, \ldots, b_{dQ}$. Notice that by considering a fixed precision given by a number of decimals, we can bound this value by 10^p, where p is the number of decimals taken into account, since every possible rational number described with a precision of p decimals can be expressed as $n/10^p$, so in the worst case $mod(10^p, d_0^Q) = mod(10^p, d_1^Q) = 0$ and hence 10^p is a multiple of both.[4]
- $R(a, b_i) = r \in [0, 1]\ \forall 1 \leqslant i \leqslant d^Q$.
- $C(b_i) = \alpha_1\ \forall 1 \leqslant i \leqslant d$ and $C(b_j) = \alpha_2\ \forall (d+1) \leqslant j \leqslant d^Q$ with $d \in \{d_0^Q, d_1^Q\}$, $\alpha_1, \alpha_2 \in \mathcal{S}(C)$, and $\alpha_1 \geqslant \alpha_2$.

With this kind of interpretations, the following cases are possible:

- $1 \geqslant r \geqslant \alpha_1$: the relative cardinality of C with respect to R_a is

$$ER(C/R_a) = (1-r)/(0/0) + (r-\alpha_1)/0 + (\alpha_1 - \alpha_2)/q + \alpha_2/1,$$

  with $q \in \{q_0^Q, q_1^Q\}$. Hence

$$D(a) = (1-r)u^Q + (r-\alpha_1)Q(0) + (\alpha_1 - \alpha_2)Q(q) + \alpha_2 Q(1),$$

  where u^Q is the value assigned to the quantifier Q in the case "undefined", i.e. $Q(0/0) = u^Q$.
- $\alpha_1 \geqslant r \geqslant \alpha_2$: the relative cardinality of C with respect to R_a is

$$ER(C/R_a) = (1-r)/(0/0) + (r-\alpha_2)/0 + \alpha_2/1.$$

  Hence

$$D(a) = (1-r)u^Q + (r-\alpha_2)Q(0) + \alpha_2 Q(1).$$

- $\alpha_2 \geqslant r \geqslant 0$: the relative cardinality of C with respect to R_a is

$$ER(C/R_a) = (1-r)/(0/0) + r/1.$$

  Hence

$$D(a) = (1-r)u^Q + rQ(1).$$

5.6.1. *Number of individuals*

From the kind of interpretations we consider, it is obvious that the number of individuals required to calculate $\mathcal{S}(D)$ is any value in the set

$$\mathbb{N}_D = \left\{n + n' \colon \left(mod\left(n, d_0^Q\right) = mod\left(n, d_1^Q\right) = 0\right) \wedge (n' \in \mathbb{N}_C)\right\}.$$

[4] For the sake of simplicity we could consider $d_0^Q = d_1^Q = d^Q = 10^p$, though this is not the lower possible value of d^Q in general.

5.6.2. *Calculating* $\mathcal{S}(D)$

From the expressions above it is clear that $\mathcal{S}(D)$ depends on Q (fixed), r, α_1 and α_2 with the restrictions $r \in [0,1]$ and $\alpha_1, \alpha_2 \in \mathcal{S}(C)$ with $\alpha_1 \geqslant \alpha_2$. Let $\mathcal{S}(C) = [\beta_C, \tau_C]$. By considering only values of a certain precision of p decimals, $\mathcal{S}(D)$ can be computed by Algorithm 1, where the round operator rounds the result to the precision of p decimals employed. The complexity of this algorithm is O(1) if we consider that the precision p is fixed. The algorithm performs $O(10^{3p})$ operations.

5.6.3. *Some particular cases*

Most of the relative quantifiers usually employed verify $Q(0), Q(1) \in \{0,1\}$. Hence, providing analytical results for these cases can help to improve the average cost of reasoning algorithms.

ALGORITHM 1. Algorithm to compute $S(D)$.

(1) Calculate $\mathcal{S}(C)$.
(2) $\tau_D = 0$, $\beta_D = 1$.
(3) For every $r \in [0,1]$ and every pair $\alpha_1, \alpha_2 \in \mathcal{S}(C)$ with $\alpha_1 \leqslant \alpha_2$, all three with precision of p decimals
  (a) If $1 \geqslant r \geqslant \alpha_1$, then for every $Q(q) \in \{0,1\}$
    (i) $v \leftarrow round((1-r)u^Q + (r-\alpha_1)Q(0) + (\alpha_1 - \alpha_2)Q(q) + \alpha_2 Q(1))$.
    (ii) If $v < \beta_D$ then $\beta_D \leftarrow v$.
    (iii) If $v > \tau_D$ then $\tau_D \leftarrow v$.
  (b) If $\alpha_1 \geqslant r \geqslant \alpha_2$,
    (i) $v \leftarrow round((1-r)u^Q + (r-\alpha_2)Q(0) + \alpha_2 Q(1))$.
    (ii) If $v < \beta_D$ then $\beta_D \leftarrow v$.
    (iii) If $v > \tau_D$ then $\tau_D \leftarrow v$.
  (c) If $\alpha_2 \geqslant r \geqslant 0$,
    (i) $v \leftarrow round((1-r)u^Q + rQ(1))$.
    (ii) If $v < \beta_D$ then $\beta_D \leftarrow v$.
    (iii) If $v > \tau_D$ then $\tau_D \leftarrow v$.

PROPOSITION 5.6. *Let* $Q(0) = 0$, $Q(1) = 1$. *Then*

$$\mathcal{S}(D) = \left[\min\left\{u^Q, \beta_C\right\}, \tau_C + (1-\tau_C)u^Q\right].$$

PROOF. We have the following cases:

- Let $1 \geqslant r \geqslant \alpha_1$. Then

$$D(a) = \begin{cases} (1-r)u^Q + \alpha_2, & Q(q) = 0, \\ (1-r)u^Q + \alpha_1, & Q(q) = 1, \end{cases}$$

  that is $D(a) = (1-r)u^Q + \alpha$ with $1 \geqslant r \geqslant \alpha$ and $\alpha \in \mathcal{S}(C)$ with $\mathcal{S}(C) = [\beta_C, \tau_C] \cup V_C$. Then for $\alpha \in [\beta_C, \tau_C]$ we have $r \in [\beta_C, 1]$ and we obtain $[\beta_C, \tau_C + (1-\tau_C)u^Q] \subseteq \mathcal{S}(D)$.
- Let $\alpha_1 \geqslant r \geqslant \alpha_2$. Then

$$D(a) = (1-r)u^Q + \alpha_2.$$

Hence $\beta_C \leqslant \alpha_2 \leqslant r \leqslant \tau_C$ and we obtain $[(1-\tau_C)u^Q+\beta_C, (1-\tau_C)u^Q+\tau_C] \subseteq \mathcal{S}(D)$.

- Let $\alpha_2 \geqslant r \geqslant 0$. Then

$$D(a) = (1-r)u^Q + r.$$

Hence $0 \leqslant r \leqslant \alpha_2 \leqslant \tau_C$ and we obtain $[u^Q, (1-\tau_C)u^Q + \tau_C] \subseteq \mathcal{S}(D)$.

In summary, from the set of values $\mathcal{S}(C) = [\beta_C, \tau_C]$ we obtain

$$\begin{aligned}
&\big[\beta_C, \tau_C + (1-\tau_C)u^Q\big] \subseteq \mathcal{S}(D),\\
&\big[(1-\tau_C)u^Q + \beta_C, (1-\tau_C)u^Q + \tau_C\big] \subseteq \mathcal{S}(D),\\
&\big[u^Q, (1-\tau_C)u^Q + \tau_C\big] \subseteq \mathcal{S}(D).
\end{aligned}$$

Hence

$$\big[\min\{u^Q, \beta_C\}, \tau_C + (1-\tau_C)u^Q\big] \subseteq \mathcal{S}(D).$$

Finally, since we have taken into account all the possible cardinalities with the maximum possible degrees, inclusion can be changed by equality in the final results. □

COROLLARY 5.2. *Quantifiers* $\exists$ *and* $\forall$ *verify*

$$\begin{aligned}
\mathcal{S}(\exists R.C) &= [0, \tau_C],\\
\mathcal{S}(\forall R.C) &= [\beta_C, 1].
\end{aligned}$$

COROLLARY 5.3. *Quantifiers* $\exists$ *and* $\forall$ *verify*

$$\begin{aligned}
\mathcal{S}(\exists R.C) &= \mathcal{S}(\neg\forall R.\neg C),\\
\mathcal{S}(\forall R.C) &= \mathcal{S}(\neg\exists R.\neg C).
\end{aligned}$$

COROLLARY 5.4. *Let* $Q(0) = 1$, $Q(1) = 0$. *Then*

$$\mathcal{S}(QR.C) = \big[1 - \big(\tau_C + (1-\tau_C)u^Q\big), 1 - \min\{u^Q, \beta_C\}\big].$$

PROOF. Immediate by the properties of *GD*, $\neg(QR.C) \equiv (\neg Q)R.C$ and taking into account that $(\neg Q)(0) = 0$ and $(\neg Q)(1) = 1$. □

PROPOSITION 5.7. *Let* $Q(0) = 0$, $Q(1) = 0$. *Then*

$$\mathcal{S}(D) = \big[0, \max\{u^Q, (1-\tau_C)u^Q + (\tau_C - \beta_C)\}\big].$$

PROOF. We have the following cases:

- Let $1 \geqslant r \geqslant \alpha_1$. Then

$$D(a) = \begin{cases} (1-r)u^Q, & Q(q) = 0,\\ (1-r)u^Q + (\alpha_1 - \alpha_2), & Q(q) = 1. \end{cases}$$

Hence for those values $\alpha \in [\beta_C, \tau_C]$ we have $r \in [\beta_C, 1]$ and we obtain $[0, (1-\beta_C)u^Q] \subseteq \mathcal{S}(D)$ (first expression) and $[0, (1-\tau_C)u^Q + (\tau_C - \beta_C)] \subseteq \mathcal{S}(D)$ (second expression, 0 obtained when $r = 1$ and $\alpha_1 = \alpha_2$), with $(1-\tau_C)u^Q + (\tau_C - \beta_C) \geqslant (1-\beta_C)u^Q$.

- Let $\alpha_1 \geqslant r \geqslant \alpha_2$ or $\alpha_2 \geqslant r \geqslant 0$. Then

$$D(a) = (1 - r)u^Q.$$

Hence $r \in [0, \tau_C]$ and we obtain $[(1 - \tau_C)u^Q, u^Q] \subseteq \mathcal{S}(D)$.

In summary, from the set of values $\mathcal{S}(C) = [\beta_C, \tau_C]$ we obtain

$$\big[0, (1 - \tau_C)u^Q + (\tau_C - \beta_C)\big] \subseteq \mathcal{S}(D),$$
$$\big[(1 - \tau_C)u^Q, u^Q\big] \subseteq \mathcal{S}(D).$$

Hence

$$\big[0, \max\big\{u^Q, (1 - \tau_C)u^Q + (\tau_C - \beta_C)\big\}\big] \subseteq \mathcal{S}(D).$$

Finally, since we have taken into account all the possible cardinalities with the maximum possible degrees, inclusion can be changed by equality in the final results. □

COROLLARY 5.5. *Let* $Q(0) = 1$, $Q(1) = 1$. *Then*

$$\mathcal{S}(QR.C) = \big[1 - \max\big\{u^Q, (1 - \tau_C)u^Q + (\tau_C - \beta_C)\big\}, 1\big].$$

PROOF. Immediate by the properties of *GD*, $\neg(QR.C) \equiv (\neg Q)R.C$ and taking into account that $(\neg Q)(0) = 0$ and $(\neg Q)(1) = 0$. □

5.7. *Disjunction*

In this section we deal with the task of calculating the satisfiability of a fuzzy concept D of the form

$$D \equiv D_1 \sqcup D_2 \sqcup \cdots \sqcup D_s.$$

It is trivial to show that if $\mathcal{S}(D_i) = [\beta_{D_i}, \tau_{D_i}]$, in general, $\mathcal{S}(D) = [\beta_D, \tau_D]$ with

$$\beta_D \geqslant \max_{i \in \{1,\ldots,s\}} \{\beta_{D_i}\},$$
$$\tau_D = \max_{i \in \{1,\ldots,s\}} \{\tau_{D_i}\}.$$

The value of β_D can only be bounded in general because it can be the case that the different D_i are not *independent*. When two concepts C and C' are not independent, it can happen that there is no interpretation $\mathcal{I}$ such that $a \in \Delta_\mathcal{I}$ with $C^\mathcal{I}(a) = \beta_C$ and $C'^\mathcal{I}(a) = \beta_{C'}$, and hence $\max\{\beta_C, \beta_{C'}\} \notin \mathcal{S}(C \sqcup C')$, so as a consequence $\beta_{C \sqcup C'} \geqslant \max\{\beta_C, \beta_{C'}\}$.

As a simple counterexample, consider the concepts A and $\neg A$. Obviously, $\mathcal{S}(A) = \mathcal{S}(\neg A) = [0, 1]$, but $\mathcal{S}(A \sqcup \neg A) = [0.5, 1]$, i.e. $\beta_{A \sqcup \neg A} = 0.5 > 0 = \max\{\beta_A, \beta_{\neg A}\}$.

5.7.1. *Calculating the lower satisfiability bound in the absence of independence*

The previous results allow us to calculate the lower satisfiability bound directly when the concepts involved in the disjunction are pairwise-independent. Now, we must provide a procedure to calculate the lower bound when independence does not hold.

A natural way to proceed is to divide the set of concepts D_i in groups of independent concepts, calculate the satisfiability of the disjunction of the concepts in each group (for which we shall provide an algorithm later) and then to combine the results.

Let $Set(D) = \{D_1 \dots D_s\}$ be the set of concepts whose disjunction defines D. Then, the groups of dependent concepts in $Set(D)$ are the elements of the quotient set $Set(D)/\bowtie = \{G_1^D, \dots, G_r^D\}$ with $r \leqslant s$. By definition, this set is a partition of the elements in $Set(D)$.

We shall denote $\sqcup_{G_i^D}$ the concept defined by the disjunction of the concepts in group G_i^D, i.e.

$$\sqcup_{G_i^D} = \sqcup_{D_j \in G_i^D} D_j.$$

With these definitions, the procedure can be formalized as follows:

(1) Calculate $Set(D)/\bowtie$, the groups of dependent concepts in $Set(D)$ (using Propositions 5.2, 5.3, and 5.4).
(2) For each $G_i^D \in Set(D)/\bowtie$, calculate the satisfiability of $\sqcup_{G_i^D}$. Let it be $\mathcal{S}(\sqcup_{G_i^D}) = [\beta_i, \tau_i]$.
(3) $\mathcal{S}(D) = [\max_i \beta_i, \max_i \tau_i]$.

The third step is immediate since it is obvious that

$$D = \sqcup_{G_1^D} \sqcup \dots \sqcup_{G_r^D}$$

and $\sqcup_{G_i^D} \not\bowtie \sqcup_{G_j^D}$ $\forall 1 \leqslant i < j \leqslant r$. However, as we mentioned before, we must provide a way to calculate the satisfiability of each $\sqcup_{G_i^D}$, since the concepts involved in it are not independent.

Finally, let us remark a very important consequence from Proposition 5.3: all the concepts appearing in a group G_i^D are either quantified or not quantified. In the two next sections we propose algorithms to deal with each case.

5.7.2. *Satisfiability of the union of dependent and non-quantified concepts*

It is well known that the satisfiability of concepts involving just atomic concepts connected by means of $\neg$, $\sqcup$, and $\sqcap$ can only be $[0, 1]$, $[0, 0.5]$ (for crisp contradictions), or $[0.5, 1]$ (for crisp tautologies).

Though in this section we are interested in calculating the satisfiability of the union of dependent and not-quantified concepts, we can provide a general set of rules that are sufficient in order to calculate the satisfiability of any not-quantified concept, i.e. any concept whose expansion involves just atomic concepts connected by $\neg$, $\sqcup$, and $\sqcap$.

Some of these rules have been described before, but it is worth to include them here again because it is possible to check for independence at the same time the satisfiability is being calculated. This means the procedure in the previous section can be simplified by creating a single group G_1^D containing the non-quantified concepts, and checking independence just between quantified concepts in order to form the rest of the groups. Then, the satisfiability of $\sqcup_{G_1^D}$ can be obtained by recursively using the following rules:

(1) If A is an atomic concept then $\mathcal{S}(A) = [0, 1]$.
(2) Let $\mathcal{S}(C) = [\beta_C, \tau_C]$. Then $\mathcal{S}(\neg C) = [1 - \tau_C, 1 - \beta_C]$.
(3) The satisfiability of $C \sqcap D$ can be obtained from $\mathcal{S}(C)$ and $\mathcal{S}(D)$ as it is shown in Table 3.

Table 3
Satisfiability of the conjunction of non-quantified concepts.

$\mathcal{S}(C \sqcap D)$		$\mathcal{S}(D)$		
		[0, 1]	[0, 0.5]	[0.5, 1]
$\mathcal{S}(C)$	[0, 1]	?	[0, 0.5]	[0, 1]
	[0, 0.5]	[0, 0.5]	[0, 0.5]	[0, 0.5]
	[0.5, 1]	[0, 1]	[0, 0.5]	[0.5, 1]

Table 4
Satisfiability of the disjunction of non-quantified concepts.

$\mathcal{S}(C \sqcup D)$		$\mathcal{S}(D)$		
		[0, 1]	[0, 0.5]	[0.5, 1]
$\mathcal{S}(C)$	[0, 1]	?	[0, 1]	[0.5, 1]
	[0, 0.5]	[0, 1]	[0, 0.5]	[0.5, 1]
	[0.5, 1]	[0.5, 1]	[0.5, 1]	[0.5, 1]

The case $\mathcal{S}(C) = \mathcal{S}(D) = [0, 1]$ can be easily solved as follows: if $C \sqcap D$ is crisp satisfiable then $\mathcal{S}(C \sqcap D) = [0, 1]$, otherwise $C \sqcap D$ is a crisp contradiction and hence $\mathcal{S}(C \sqcap D) = [0, 0.5]$.

(4) The satisfiability of $C \sqcup D$ can be obtained from $\mathcal{S}(C)$ and $\mathcal{S}(D)$ as it is shown in Table 4.

Again, the case $\mathcal{S}(C) = \mathcal{S}(D) = [0, 1]$ can be easily solved as follows: if $\neg(C \sqcup D)$ is crisp satisfiable then $\mathcal{S}(C \sqcup D) = [0, 1]$, otherwise $C \sqcup D$ is a crisp tautology and hence $\mathcal{S}(C \sqcup D) = [0.5, 1]$.

5.7.3. *Satisfiability of the union of dependent and quantified concepts*

Let $D \equiv \sqcup_{G_k^D} \equiv Q_1 R.C_1 \sqcup Q_2 R.C_2 \sqcup \cdots \sqcup Q_t R.C_t$, where $Q_1, \ldots, Q_t$ are quantifiers and $\{C_1, \ldots, C_t\}$ is a group of dependent concepts, i.e. $C_i \bowtie C_j \ \forall 1 \leqslant i < j \leqslant t$. Let $D_i = Q_i R.C_i$. We consider now the problem of deciding the minimum extent β_D to which D is satisfiable. We already know the maximum extent τ_D to which D is satisfiable, and also that $\beta_D \geqslant \bar{\beta}_D$ with $\bar{\beta}_D = \max_{i \in \{1,\ldots,t\}} \{\beta_{D_i}\}$.

By definition, given an individual $a \in \Delta$,

$$
\begin{aligned}
\beta_D &= \inf_{\mathcal{I}} \max_i \left\{ D_i^{\mathcal{I}}(a) \right\} \\
&= \inf_{\mathcal{I}} \max_i \left\{ \int_0^1 Q_i\left(\left| \left\{ b \in \Delta^{\mathcal{I}} \colon\ R(a,b) \geqslant \tau \wedge C_i(b) \geqslant \tau \right\} \right| \right) \mathrm{d}\tau \right\} \\
&= \inf_{\mathcal{I}} \max_i \left\{ \sum_j (\alpha_j - \alpha_{j+1}) Q_i\left(\left| (R_a \cap C_i)_{\alpha_j} \right| \right) \right\}.
\end{aligned}
$$

We solve this problem by constructing an interpretation $\mathcal{I}^*$ which minimizes $\max_i\{D_i^{\mathcal{I}^*}(a)\}$. For this to be true, all the following inequalities must hold:

$$\begin{cases} D_1^{\mathcal{I}}(a) \leqslant \sigma, \\ \vdots \\ D_t^{\mathcal{I}}(a) \leqslant \sigma, \end{cases}$$

where $\sigma = \beta_D$, as well as all the TVBs in the expanded ABox, i.e. the ABox obtained after eliminating the TBox.

We are able to construct an interpretation $\mathcal{I}$ which satisfies all the above inequalities as well as all the TVBs in the expanded ABox for a given σ, provided there exists one. This could be done by means of Algorithm 2. If the algorithm has exhausted all guesses at step 2, then it must be concluded that the conjunction is not satisfiable to the degree σ.

However, we do not know the value of σ in advance, except that it must lie above the maximum of the β_{D_i}; therefore, we shall perform a dichotomic search on the values of σ by using Algorithm 3.

ALGORITHM 2. Building an interpretation that satisfies the inequalities and all the TVBs in the expanded ABox for a given σ.

(1) Determine the number n_D of individuals that must be included in the interpretation as fillers of role R, applying that

$$\mathbb{N}_{D_i \sqcup D_j} = \mathbb{N}_{D_i} \cap \mathbb{N}_{D_j}.$$

Let us remark that $\mathbb{N}_{D_i \sqcup D_j} \neq \emptyset$ since in the restrictions on the number of individuals we know that at least 10^p is a common value for all relative quantifiers (and any multiple of 10^p will be also in this set), and restrictions for absolute quantifiers take always the form "at least n individuals".

(2) "Guess" truth values for $R(a, b_k)$ and $C_i(b_k)$: these latter are actually calculated from the definition of C_i, after guessing the value of $A(b_k)$ for all primitive concepts A occurring in the definition of C_i; or recursively if C_i contains quantified concepts.

(3) Check whether any TVB in the ABox or any of the t inequalities is violated. If so, backtrack to the previous step; otherwise $\mathcal{I}$ is an interpretation satisfying the conjunction to the degree σ.

ALGORITHM 3. Performing a dichotomic search on the values of σ.

(1) $\sigma \leftarrow \bar{\beta}_D$, $\beta_D \leftarrow \tau_D$;

(2) By using Algorithm 2 try to construct an interpretation $\mathcal{I}$ that satisfies all the inequalities as well as all the TVBs in the ABox; if that is possible,

$$\beta_D \leftarrow \sigma;$$
$$\sigma \leftarrow \frac{\sigma + \bar{\beta}_D}{2};$$

otherwise,

$$\bar{\beta}_D \leftarrow \sigma;$$
$$\sigma \leftarrow \frac{\sigma + \tau_D}{2};$$

(3) if $\bar{\beta}_D - \sigma < 10^{-p}$, stop;
(4) go back to step 2.

It is easy to verify that this algorithm always stops with the correct value of β_D, up to a precision 10^{-p}, which represents the precision to which the results are needed. Furthermore, it always stops after at most $\log_2 10^p$ iterations. Although we do not provide here a formal proof for that, it can be proven that Algorithm 3 runs in polynomial space.

5.8. *Conjunction*

The satisfiability of a conjunction can be solved easily since a conjunction can be turn into the negation of a disjunction by using De Morgan's laws, and we have provided a general procedure for disjunction in the previous section.

5.9. *Summary*

We can summarize these results as follows:

- If A is an atomic concept, then $\mathcal{S}(A) = [0, 1]$.
- (Negation) If $\mathcal{S}(D) = [\beta_D, \tau_D]$, then $\mathcal{S}(\neg D) = [1 - \tau_D, 1 - \beta_D]$. This verifies $\mathcal{S}(\neg\neg D) = \mathcal{S}(D)$ and:
  - if A is atomic, $\mathcal{S}(\neg A) = \mathcal{S}(A) = [0, 1]$;
  - if $C = C_1 \sqcap \cdots \sqcap C_r$, then $\mathcal{S}(\neg C) = \mathcal{S}(\neg C_1 \sqcup \cdots \sqcup \neg C_r)$;
  - if $C = C_1 \sqcup \cdots \sqcup C_r$, then $\mathcal{S}(\neg C) = \mathcal{S}(\neg C_1 \sqcap \cdots \sqcap \neg C_r)$;
  - if $D \equiv QR.C$,
$$\mathcal{S}(\neg D) = \mathcal{S}\big(\neg(QR.C)\big) = \mathcal{S}\big((\neg Q)R.C\big). \tag{15}$$
- Let Q be an absolute quantifier such that $\text{core}(Q) \neq \emptyset$ and $\mathbb{N}\backslash\text{supp}(Q) \neq \emptyset$. If $D \equiv QR.C$ and $\mathcal{S}(C) = [\beta_C, \tau_C]$, $\mathcal{S}(D) = \mathcal{S}(QR.C) = [\beta_D, \tau_D]$, with
$$\beta_D = (1 - \tau_C)Q(0), \tag{16}$$
$$\tau_D = \max\big\{\big(\tau_C + (1 - \tau_C)Q(0)\big), Q(0)\big\}. \tag{17}$$
- Let Q be a relative quantifier such that $\text{core}(Q) \neq \emptyset$ and $[0, 1] \cap \mathbb{Q}\backslash\text{supp}(Q) \neq \emptyset$ with $u^Q = Q(x/0)$ (i.e. u^Q is the value returned by the quantifier when the relative cardinality is undefined). If $D \equiv QR.C$ and $\mathcal{S}(C) = [\beta_C, \tau_C]$,
  - If $Q(0) = 0$ and $Q(1) = 1$,
$$\mathcal{S}(D) = \big[\min\big\{u^Q, \beta_C\big\}, \tau_C + (1 - \tau_C)u^Q\big].$$
    In particular, for quantifiers $\exists$ and $\forall$ we have $u^\exists = 0$, $u^\forall = 1$, and
$$\mathcal{S}(\exists R.C) = [0, \tau_C],$$
$$\mathcal{S}(\forall R.C) = [\beta_C, 1].$$
  - If $Q(0) = 1$ and $Q(1) = 0$,
$$\mathcal{S}(QR.C) = \big[1 - \big(\tau_C + (1 - \tau_C)u^Q\big), 1 - \min\big\{u^Q, \beta_C\big\}\big].$$

– If $Q(0) = 0$ and $Q(1) = 0$,

$$\mathcal{S}(D) = \big[0, \max\{u^Q, (1-\tau_C)u^Q + (\tau_C - \beta_C)\}\big].$$

– If $Q(0) = 1$ and $Q(1) = 1$,

$$\mathcal{S}(QR.C) = \big[1 - \max\{u^Q, (1-\tau_C)u^Q + (\tau_C - \beta_C)\}, 1\big].$$

The remaining cases are solved by means of Algorithm 1 in O(1) with a fixed precision of p decimals.

- If $D \equiv D_1 \sqcup D_2 \sqcup \cdots \sqcup D_s$ with $\mathcal{S}(D_i) = [\beta_{D_i}, \tau_{D_i}]$, then $\mathcal{S}(D) = [\beta_D, \tau_D]$, with

$$\beta_D \geqslant \max_{i\in\{1,\ldots,s\}} \{\beta_{D_i}\},$$
$$\tau_D = \max_{i\in\{1,\ldots,s\}} \{\tau_{D_i}\}.$$

In particular, if the fuzzy concepts D_i are pairwise independent,

$$\beta_D = \max_{i\in\{1,\ldots,s\}} \{\beta_{D_i}\},$$
$$\tau_D = \max_{i\in\{1,\ldots,s\}} \{\tau_{D_i}\}.$$

The value of β_D with a precision degree of p decimals, when the D_i are not independent, is obtained by means of Algorithm 3 that performs a dichotomic search after guessing values of the atomic concepts and roles that appear in the D_i's.

- The satisfiability of conjunctions can be obtained by using De Morgan's laws and the latter result.

6. Conclusions

$\mathcal{ALCQ}_F^+(\mathsf{D})$ allows for concept description involving fuzzy linguistic quantifiers of the absolute and relative kind, and using qualifiers. In this chapter we have introduced algorithms to perform two important reasoning tasks with this logic: reasoning about instances, using the semantics of $\mathcal{ALCQ}_F^+(\mathsf{D})$, and calculating the fuzzy satisfiability of a fuzzy concept, for which a polynomial-space algorithm is provided. Though infinite interpretations are taken into account in the definition of the semantics of the language $\mathcal{ALCQ}_F^+(\mathsf{D})$, in practice and due to the physical limitations of computers, we have considered a finite number of individuals. Since the same limitations put a bound on the number of different membership degrees we can deal with, we calculate the fuzzy satisfiability of a fuzzy concept up to a certain precision degree, given as a number of decimals p. In fact, we believe a precision of one decimal digit is already enough for most applications and a precision of more than two decimal digits would be absolutely unnecessary in all applications.

In addition to the general algorithm, we have obtained some results showing that we can calculate the fuzzy satisfiability of a fuzzy concept directly in some (the most common) cases, hence improving the performance of the reasoning process. Some of these results are based on a result about *independence* of fuzzy concepts that we also introduce as a remarkable result.

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