Second-order criterion for blind source extraction

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A second-order criterion for blind signal extraction in instantaneous linear mixtures has recently been proposed. It is proved that, with an adequate choice of autocorrelation time lags, the criterion leads indeed to a successful source extraction in the noiseless case. Using this criterion, the source identifiability conditions turn out to be the same as in the popular second-order blind identification method for blind source separation.

Introduction: Recently, a novel technique for blind source extraction (BSE), relying on second-order statistics (SOS), has been presented [1]. The technique is based on the instantaneous linear mixing model for the observed signals $\boldsymbol{x}(t) = [x_1(t), \ldots, x_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$:

$$\boldsymbol{x}(t) = \boldsymbol{M}\boldsymbol{s}(t) \tag{1}$$

where $s(t) = [s_1(t), \ldots, s_N(t)]^{\mathrm{T}} \in \mathbb{R}^N$ is the source vector and $\boldsymbol{M} = [\boldsymbol{m}_1, \ldots, \boldsymbol{m}_N] \in \mathbb{R}^{N\times N}$ represents the full column rank mixing matrix. BSE aims at estimating one of the sources at the extractor output:

$$y(t) = \boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}(t) \tag{2}$$

through an appropriately designed extracting vector $\boldsymbol{n} \in \mathbb{R}^N$. Let $\boldsymbol{R}_\tau = \mathrm{E}\{\boldsymbol{x}(t)\boldsymbol{x}^{\mathrm{T}}(t-\tau)\}$ denote the sensor-output autocorrelation matrix at time lag τ. It is claimed in [1] that a valid extracting vector $\boldsymbol{n}$ can be obtained from the minimisation of functional

$$J(\boldsymbol{n}, \boldsymbol{t}, \boldsymbol{d}) = \sum_{k=0}^{K} \|\boldsymbol{R}_{\tau_k}\boldsymbol{n} - d_{\tau_k}\boldsymbol{t}\|^2 \tag{3}$$

with respect to vectors $\boldsymbol{n}, \boldsymbol{t} \in \mathbb{R}^N$ and $\boldsymbol{d} = [d_{\tau_0}, d_{\tau_1}, \ldots, d_{\tau_K}]^{\mathrm{T}} \in \mathbb{R}^{K+1}$. In that work, this claim is given a geometrical interpretation in terms of oblique projection operators and demonstrated through numerical experiments. The sources are assumed to be uncorrelated and coloured, but no evidence is presented as to why their spectra should be distinct. This Letter provides a more thorough justification for this approach and proves that, under the assumptions of model (1) and with an appropriate choice of time lags $\{\tau_k\}_{k=0}^K$, the minimisation of (3) is indeed achieved if and only if $\boldsymbol{n}$ is a valid extracting vector. In addition, the source identifiability conditions are found to be the same as in the well-known second-order blind identification (SOBI) method [2] for blind source separation (BSS).

Any valid extracting vector minimises the criterion: Let us assume that $\boldsymbol{n}$ is a valid extraction vector for source $s_i(t)$. Then, by definition:

$$\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}(t) = \alpha s_i(t) \tag{4}$$

for an admissible (but otherwise irrelevant) non-zero scale factor $\alpha \in \mathbb{R}$. Left-multiplying both sides of (4) by $\boldsymbol{x}^{\mathrm{T}}(t-\tau)$, taking mathematical expectations and exploiting the source uncorrelation assumption, one arrives at $\boldsymbol{R}_\tau\boldsymbol{n} = \alpha r_i(\tau)\boldsymbol{m}_i$, where $r_i(\tau)$ stands for the ith source autocorrelation function at time lag τ. Hence, all valid extracting vectors $\boldsymbol{n}$ for source $s_i(t)$ exactly minimise function (3) for any τ with $d_\tau\boldsymbol{t} = \alpha r_i(\tau)\boldsymbol{m}_i$. This result provides supporting evidence for the validity of BSE criterion (3). Next, we see that, under some additional conditions, all minimisers of (3) are indeed valid extracting vectors.

Any minimiser of the criterion is a valid extracting vector: Let $\boldsymbol{R}_\tau^s = \mathrm{E}\{s(t)s^{\mathrm{T}}(t-\tau)\} = \mathrm{diag}(r_1(\tau), r_2(\tau), \ldots, r_N(\tau))$ denote the source autocorrelation matrix at time lag τ. Assume that vectors $\boldsymbol{n}$, $\boldsymbol{t}$ and $\boldsymbol{d}$ are non-trivial exact minimisers of (3) for a time-lag set $\mathrm{T} = \{\tau_k\}_{k=0}^K$. By virtue of model (1), we have

$$\boldsymbol{R}_{\tau_k}\boldsymbol{n} = \boldsymbol{M}\boldsymbol{R}_{\tau_k}^s\boldsymbol{M}^{\mathrm{T}}\boldsymbol{n} = \boldsymbol{M}\boldsymbol{R}_{\tau_k}^s a, \quad 0 \le k \le K \tag{5}$$

where

$$\boldsymbol{a} = \boldsymbol{M}^{\mathrm{T}}\boldsymbol{n}, \quad [\boldsymbol{a}]_i = a_i = \boldsymbol{m}_i^{\mathrm{T}}\boldsymbol{n} \tag{6}$$

Extractor output (2) can then be written as

$$y(t) = \sum_{i=1}^{N} a_i s_i(t) \tag{7}$$

Hence, a successful source extraction requires that at most one entry of $\boldsymbol{a}$ be different from zero.

Since the triplet $(\boldsymbol{n}, \boldsymbol{t}, \boldsymbol{d})$ is a perfect minimiser of criterion (3), the last term in (5) can be expressed as

$$\boldsymbol{M}\boldsymbol{R}_{\tau_k}^s\boldsymbol{a} = d_{\tau_k}\boldsymbol{t} \quad 0 \le k \le K \tag{8}$$

As $\boldsymbol{M}$ is full column rank, its columns span $\mathbb{R}^N$ and there is a unique linear combination, with coefficients stacked in vector $\boldsymbol{b} \in \mathbb{R}^N$, yielding $\boldsymbol{t}$:

$$\boldsymbol{M}\boldsymbol{b} = \boldsymbol{t} \tag{9}$$

Combining (8) and (9), we must have that $\boldsymbol{R}_{\tau_k}{}^s\,\boldsymbol{a} = d_{\tau_k}\boldsymbol{b}$, which implies that

$$a_i r_i(\tau_k) = d_{\tau_k} b_i, \quad 1 \le i \le N, \quad 0 \le k \le K \tag{10}$$

Define the ith-source autocorrelation vector as $\boldsymbol{r}_i(\mathrm{T}) = [r_i(\tau_0), r_i(\tau_1), \ldots, r_i(\tau_K)]^{\mathrm{T}}$. Then, relationship (10) can be compactly expressed as $\boldsymbol{r}_i(\mathrm{T})a_i = \boldsymbol{d}b_i$, $1 \le i \le N$. Since $\boldsymbol{r}_i(\mathrm{T})$ and $\boldsymbol{d}$ are non-null vectors, we can only have either $a_i = b_i = 0$ or $\boldsymbol{r}_i(\mathrm{T}) = \boldsymbol{d}b_i/a_i$. If vector $\boldsymbol{a}$ had another element different from zero, say its jth entry, $j \ne i$, then we would also have $\boldsymbol{r}_j(\mathrm{T}) = \boldsymbol{d}b_j/a_j$, and the autocorrelation functions of $s_i(t)$ and $s_j(t)$ would be proportional to each other at the selected lags, $\boldsymbol{r}_i(\mathrm{T}) = c_{ij}\boldsymbol{r}_j(\mathrm{T})$, with $c_{ij} = b_i a_j/(b_j a_i)$. To prevent this possibility, which leads to a non-extracting solution [recall (7)], it is necessary to choose a time-lag set T such that $\boldsymbol{r}_i(\mathrm{T})$ and $\boldsymbol{r}_j(\mathrm{T})$ are not parallel, $1 \le i < j \le N$. If this condition is met, there can be at most one coefficient $a_i \ne 0$. From (6), it follows that $\boldsymbol{m}_j^{\mathrm{T}}\boldsymbol{n} = 0$, $j \ne i$, and, according to (2) and (7), the application of vector $\boldsymbol{n}$ verifying such constraints yields $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}(t) = \alpha s_i(t)$, with $\alpha = a_i = \boldsymbol{m}_i^{\mathrm{T}}\boldsymbol{n}$. Therefore, vector $\boldsymbol{n}$ obtained from the exact minimisation of criterion (3) is a valid extracting vector as long as the time-lag set T fulfils the above condition.

In general, the finite sample size or the presence of noise will prevent functional (3) from being cancelled exactly. Nevertheless, its minimisation constitutes a somewhat natural least squares (LS) criterion, the solutions of which are thus expected to lie near valid extractors.

Source identifiability conditions: According to the above proof, a time-lag set for which a pair of source autocorrelation vectors are parallel does not guarantee source identifiability through the minimisation of criterion (3), even if the source spectra are different. At first sight, this condition may seem slightly more stringent than the uniqueness condition of the SOBI algorithm for BSS [2]: $\forall 1 \le i < j \le N$, $\exists k$, $0 \le k \le K$, such that $r_i(\tau_k) \ne r_j(\tau_k)$, which in our notation can be expressed as $\boldsymbol{r}_i(\mathrm{T}) \ne \boldsymbol{r}_j(\mathrm{T})$, $1 \le i < j \le N$. However, the data whitening step in SOBI enforces $r_i(0) = 1$, $1 \le i \le N$, so that two source autocorrelation vectors can only be parallel if they are identical. Hence, the necessary conditions for source identifiability are actually the same in both techniques. As in SOBI, increasing the number of lags will also reduce the probability of degeneracy in the BSE criterion (3). Asymptotically, as the number of lags tends to infinity, the condition becomes that no pair of source spectra be equal up to scale. Again, owing to the amplitude constraints imposed by prewhitening, this asymptotic condition is identical to SOBI's.

Conclusions: This Letter has proven that the minimisation of function (3) is a valid criterion for BSE under the same conditions as the well-known SOBI technique for BSS. It should be noted, however, that the global convergence of the alternating LS algorithm proposed in [1] to minimise the criterion is not guaranteed. The analysis of this iterative algorithm should be addressed in future investigations.

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