UNIFIED FORMULATION OF CLOSED-FORM ESTIMATORS FOR BLIND SOURCE SEPARATION IN REAL INSTANTANEOUS LINEAR MIXTURES

Vicente Zarzoso and Asoke K. Nandi

Signal Processing and Communications Group, Department of Electrical Engineering and Electronics, The University of Liverpool, Brownlow Hill, Liverpool L69 3GJ, England, UK
Tel/Fax: +44 151 794 4525/4540, e-mail: {vicente, aknandi}@liv.ac.uk
web: http://www.liv.ac.uk/~{vicente, aknandi}

ABSTRACT

This contribution provides a unified framework for the analytic or closed-form estimators for blind separation of independent source signals in real-valued instantaneous linear mixtures, in the noiseless two-source two-sensor scenario. First, the connections among three existing 4th-order analytic formulae (CF, AML and EML) are clarified. Next, a general expression for the estimation of the relevant separation parameter from the data nth-order statistics is unveiled, of which the EML estimator is a particular case at $n = 4$, and from which a novel third-order estimator is derived at $n = 3$. Asymptotic performance analysis results are also presented. Associated contrast-function optimization criteria are shown to extend the applicability of a known 4th-order contrast function in the two-signal case. Simulations illustrate and validate the theoretical exposition.

1. INTRODUCTION

The present paper addresses the *blind source separation (BSS)* problem, in the context where the observed mixtures $\mathbf{y} = [y_1, \ldots, y_p]^{\mathrm{T}} \in \mathbb{R}^p$ of the unknown statistically independent source signals $\mathbf{x} = [x_1, \ldots, x_q]^{\mathrm{T}} \in \mathbb{R}^q$ can be considered as noiseless, linear, instantaneous and real-valued (symbol $^{\mathrm{T}}$ stands for the transposition operator). In such a case, the mixture can be represented by an unknown mixing matrix $M \in \mathbb{R}^{p\times q}$, so that the underlying model reads:

$$\mathbf{y} = M\mathbf{x}. \tag{1}$$

The goal consists of recovering the sources and the mixing matrix from the observed mixtures. After pre-whitening the observation vector (comprising 2nd-order decorrelation plus power normalization), a whitened vector $\mathbf{z} \in \mathbb{R}^q$ is obtained, which is related to the unit-variance sources through an orthogonal transformation Q [12]:

$$\mathbf{z} = Q\mathbf{x}. \tag{2}$$

In the two-source two-sensor case, the orthogonal matrix Q becomes an elementary Givens rotation:

$$Q = \mathcal{Q}(\theta), \qquad \text{with}$$
$$\mathcal{Q}(\cdot):\ \mathbb{R} \mapsto \mathbb{R}^{2\times 2}, \quad \mathcal{Q}(\theta) \triangleq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{3}$$

This work is supported by The University of Liverpool.

The estimation of the mixing matrix, and in turn the source extraction, is hence reduced to the estimation of unknown parameter θ. The characteristics of this simplified scenario are indeed restrictive. However, a clear understanding of the problem in this basic set-up is of primal importance, since the solution in more elaborate environments (such as more than two signals, additive measurement noise, etc.) may benefit from results on this fundamental case [2,12].

We focus on the methods whereby the relevant parameter is estimated in analytic or closed-form, as opposed to techniques requiring iterative optimization (e.g., [5]) or other computationally demanding procedures. Essentially, closed-form solutions obtain the angle estimate directly by means of a formula composed of certain whitened-vector statistics. Specific methods differ in the statistics and formulae used. In [1], Comon first proved that this analytic estimation is indeed feasible in the two-signal case and, by nulling an output 4th-order cumulant, he suggested:

$$\mathrm{tg}\,\hat\theta_{\mathrm{CF}} = -\rho/2 + \mathrm{sign}(\rho)\sqrt{\rho^2/4+1}, \qquad \rho \triangleq \frac{\kappa_{31}^z - \kappa_{13}^z}{\kappa_{22}^z}, \tag{4}$$

which we refer to as *Comon's formula (CF)*. The terms $\kappa_{mn}^z \triangleq \mathrm{Cum}_{mn}[z_1, z_2]$ denote the $(m+n)$th-order cumulants [8,12] of the whitened signals. Applying the maximum likelihood (ML) principle and the Gram-Charlier expansion of the source probability density function (pdf), the *approximate ML (AML)* estimator was derived in [4]:

$$\mathrm{tg}(4\hat\theta_{\mathrm{AML}}) = \frac{\mathrm{E}[r^4\sin\phi]}{\mathrm{E}[r^4\cos\phi]} = \frac{4(\kappa_{31}^z - \kappa_{13}^z)}{\kappa_{40}^z - 6\kappa_{22}^z + \kappa_{04}^z},$$
$$r\,\mathrm{e}^{j\phi} = z_1 + jz_2, \qquad j \triangleq \sqrt{-1}. \tag{5}$$

The AML was generalized to scenarios outside the restrictions of its original definition [with the much weaker condition that the source kurtosis sum (sks) be different from zero] via the *extended ML (EML)* estimator [10]:

$$\hat\theta_{\mathrm{EML}} = \frac{1}{4}\arg\big(\xi\cdot\mathrm{sign}(\gamma')\big), \tag{6}$$

with

$$\left\{\begin{array}{l} \xi = \mathrm{E}[r^4\mathrm{e}^{j4\phi}] = (\kappa_{40}^z - 6\kappa_{22}^z + \kappa_{04}^z) + j4(\kappa_{31}^z - \kappa_{13}^z) \\ \gamma' = \mathrm{E}[r^4] - 8 = \kappa_{40}^z + 2\kappa_{22}^z + \kappa_{04}^z, \end{array}\right. \tag{7}$$

In *Proceedings ICASSP*, Istanbul, Turkey, June 5-9, 2000, Vol. V, pp. 3160-3163.

where γ' is an estimate of the sks $\gamma \triangleq \kappa_{40}^x + \kappa_{04}^x$.

The goal of this contribution is to establish the connections among these closed-form solutions and to present new results on the topic. Altogether, the real-mixture BSS problem is endowed with a novel unifying framework in the context of its analytic solutions.

2. CONNECTIONS

An equivalent expression for the CF, which we thus call *alternative CF (ACF)*, is given in [6]:

$$\mathrm{tg}(2\hat{\theta}_{\mathrm{ACF}}) = \frac{2}{\rho}. \tag{8}$$

Straightforward application of the double-angle tangent formula proves the equivalence between eqns. (4) and (8). Such formula and some simplifications lead to the AML estimator (5). However, a particular relationship among whitened 4th-order cumulants [1] is necessary to arrive from (8) to (5), which only holds for infinite sample size (perfect cumulant estimation).

For finite (although large enough) sample size, we can relate the estimators' variances. From (4) and (8), the ACF can be restated as the argument of the complex statistic (centroid) $\xi_{\mathrm{ACF}} = (\kappa_{31}^z - \kappa_{13}^z) + j2\kappa_{22}^z$. Following the methodology used in [10] for the analysis of the EML estimator, together with the Gaussian asymptotic approximation developed in [13], we reach:

$$\sigma_{\mathrm{EML}}^2 = \frac{\mu_{60}^x + \mu_{06}^x - 2\mu_{40}^x\mu_{04}^x}{T(\kappa_{40}^x + \kappa_{04}^x)^2}, \tag{9}$$

and

$$\sigma_{\mathrm{CF}}^2 = \sigma_{\mathrm{EML}}^2 + \Delta\sigma^2(\theta), \quad \text{with} \quad \Delta\sigma^2(\theta) = \frac{4}{T\gamma^2 \mathrm{tg}\, 2\theta} \cdot \left\{ \frac{(\mu_{40}^x\mu_{04}^x - 1)}{\mathrm{tg}\, 2\theta} + (\mu_{50}^x\mu_{03}^x - \mu_{30}^x\mu_{05}^x) \right\}, \tag{10}$$

σ_{EML}^2 and σ_{CF}^2 being the asymptotic variances of the EML and CF estimators, respectively, γ the sks, T the sample size and $\mu_{mn}^x \triangleq \mathrm{E}[x_1^m x_2^n]$ the source $(m+n)$th-order moments. The CF performance is hence dependent on the unknown parameter θ, as opposed to the EML, which is orthogonal invariant [10, 13]. We have that $\sigma_{\mathrm{CF}}^2 \geqslant \sigma_{\mathrm{EML}}^2$, $\forall\theta$, when there is a symmetric source or both sources have identical distribution (actually it is only required that they have equal 3rd- and 5th-order moments). For θ close to zero the CF quality severely degrades, and a considerably higher variance than the EML's is expected.

3. A CLOSED-FORM ESTIMATION FAMILY

It is now shown that the EML is actually a member of a wider family of estimators. The family is characterized by the following result.

Theorem 1. *Define $\xi_n(\mathbf{z})$ as the following complex weighted sum of pairwise nth-order cumulants of the components of $\mathbf{z}$, with $n \in \mathbb{N}^+$:*

$$\xi_n(\mathbf{z}) \triangleq \sum_{r=0}^{n} \binom{n}{r} j^r \kappa_{n-r,\,r}^z. \tag{11}$$

If $\mathbf{z} = \mathcal{Q}(\theta)\,\mathbf{x}$, with $\mathbf{x}$ made up of independent components, then

$$\xi_n(\mathbf{z}) = \mathrm{e}^{jn\theta}\xi_n(\mathbf{x}), \tag{12}$$

where, according to (11), $\xi_n(\mathbf{x}) = \kappa_{n0}^x + j^n\kappa_{0n}^x$.

Sketch of the proof. The proof relies on the multilinearity property of cumulants, the source independence assumption, the orthogonal nature of the remaining mixing transformation (3), Newton's binomial expansion, Euler's formula, and simple algebraic manipulations. ∎

In the light of Theorem 1, the hidden rotation can be estimated in analytic form as:

$$\hat{\theta}_n = \frac{1}{n}\arg\left(\frac{\xi_n(\mathbf{z})}{\xi_n(\mathbf{x})}\right). \tag{13}$$

In a genuine blind problem the source statistics, and hence $\xi_n(\mathbf{x})$, are unknown. But in some situations $\xi_n(\mathbf{x})$ can be estimated from the available data, as illustrated next.

<u>Case $n=2$</u>. Due to the unit-variance source convention, $\xi_2(\mathbf{z}) = 0$, so no meaningful estimation can be accomplished by (13) at this order. Indeed, all 2nd-order information is depleted after pre-whitening so that no 2nd-order based technique is able to recover the missing orthogonal transformation if the time information is disregarded or cannot be exploited (as in the i.i.d. case) [12].

<u>Case $n=3$</u>. The source 3rd-order statistic $\gamma_3 \triangleq \xi_3(\mathbf{x}) = (\kappa_{30}^x - j\kappa_{03}^x)$ may indirectly be obtained through:

$$\gamma_3' \triangleq \kappa_{21}^z + j\kappa_{12}^z = \tfrac{1}{2}\sin 2\theta\, \mathrm{e}^{j\theta}(\kappa_{30}^x - j\kappa_{03}^x), \tag{14}$$

so that θ may be estimated at this order via:

$$\hat{\theta}_3 = \frac{1}{2}\arg\left(\frac{\xi_3(\mathbf{z})}{\gamma_3'}\right), \tag{15}$$

selecting $\hat{\theta}_3 = 0$ if $\gamma_3' = 0$. This novel expression is called *third-order blind signal estimator (TOBSE)*, and is applicable provided there is at least one asymmetric source. Under this condition, TOBSE may prove advantageous relative to the 4th-order procedures, given the higher accuracy of 3rd-order cumulant estimates. Remark that TOBSE statistics may also be computed from the whitened scatter-plot points in polar form, as for the AML and EML methods:

$$\begin{cases} \xi_3(\mathbf{z}) = \mathrm{E}[(z_1 + jz_2)^3] = \mathrm{E}[r^3\mathrm{e}^{j3\phi}] \\ \hat{\gamma}_3 = \mathrm{E}[z_1 z_2(z_1 + jz_2)] = \mathrm{E}[\frac{1}{2}r^3\mathrm{e}^{j\phi}\sin 2\phi]. \end{cases} \tag{16}$$

The asymptotic performance analysis of TOBSE can be carried out along the lines of [10, 13]. After some simplifications, one can arrive at the approximate asymptotic variance:

$$\begin{aligned} \sigma_{\mathrm{TOBSE}}^2 &= \frac{\tilde{\sigma}^2}{9T|\gamma_3|^4}, \qquad \text{with} \\ \tilde{\sigma}^2 &= (\kappa_{30}^x)^2(\mu_{06}^x + 9\mu_{40}^x - 6\mu_{04}^x) \\ &\quad + (\kappa_{03}^x)^2(\mu_{60}^x + 9\mu_{04}^x - 6\mu_{40}^x) - 20(\kappa_{30}^x\kappa_{03}^x)^2. \end{aligned} \tag{17}$$

<u>Case $n=4$</u>. At 4th-order, expression (11) becomes the EML centroid ξ [eqn. (7)]. The lack of prior knowledge on the source distribution is overcome through the estimation of the sks γ via γ' (7). Since $\gamma' \in \mathbb{R}$, only its sign is of interest for the estimation of θ. The asymptotic variance of the EML estimator appears in eqn. (9) [10, 13].

4. OPTIMALITY PRINCIPLES

4.1. General Criteria

Denote as $\mathbf{s}$ the signal vector at the separator output, given by $\mathbf{s} = \mathcal{Q}(-\hat{\theta})\mathbf{z}$.

Theorem 2. *The family of estimators given analytically by equation (13) are the local minimizers of*

$$J_{\mathrm{m}n}(\hat{\theta}) = \left|\xi_n(\mathbf{s}) - \xi_n(\mathbf{x})\right|^2, \tag{18}$$

or, equivalently, the local maximizers of

$$J_{\mathrm{M}n}(\hat{\theta}) = \mathrm{Re}\big(\xi_n(\mathbf{x})^*\xi_n(\mathbf{s})\big), \tag{19}$$

in which symbol * *represents complex conjugation.*

Sketch of the proof. That the minimization of $J_{\mathrm{m}n}$ and the maximization of $J_{\mathrm{M}n}$ are equivalent follows from $\xi_n(\mathbf{s}) = \xi_n(\mathbf{z})\mathrm{e}^{-jn\hat{\theta}}$ and $|\xi_n(\mathbf{z})| = |\xi_n(\mathbf{x})| =$ constant for a given pair of sources. Imposing the necessary local maximization conditions $\partial J_{\mathrm{M}n}/\partial\hat{\theta} = 0$ and $\partial^2 J_{\mathrm{M}n}/\partial\hat{\theta}^2 < 0$ leads to solutions (13). ■

Minimization of (18) corresponds to a cumulant matching criterion whereby the mismatch between a linear combination of output nth-order cumulants ($\xi_n(\mathbf{s})$) and the same linear combination of source cumulants ($\xi_n(\mathbf{x})$) is minimized. Maximization of (19) yields a contrast function optimization criterion. According to Theorem 2, the optimization of these equivalent criteria is solved in closed-form by (13).

4.2. Contrast Function at Fourth Order

At $n = 4$, eqn. (19) corresponds to the contrast function of the EML estimator. This settles the question of whether an optimization criterion existed for this estimator [11]. From (11), this contrast function can be written as:

$$J_{\mathrm{EML}} = J_{\mathrm{M}4} = \gamma\,(\kappa_{40}^s - 6\kappa_{22}^s + \kappa_{04}^s). \tag{20}$$

Straightforward cumulant differentiation shows that J_{EML} above and the functional

$$\tilde{J}_{\mathrm{EML}} = \varepsilon_\gamma\,(\kappa_{40}^s + \kappa_{04}^s), \qquad \text{with } \varepsilon_\gamma = \mathrm{sign}(\gamma), \tag{21}$$

have identical local extrema [13]. It follows that the local maxima of $\tilde{J}_{\mathrm{EML}}$ are given in closed-form by (13).

On the other hand, a 4th-order contrast function is proposed in [7], which is valid when all the sources have the same sign of kurtosis. In the two-signal case, this function becomes:

$$I_4(\mathbf{s}) = \varepsilon(\kappa_{40}^s + \kappa_{04}^s), \qquad \varepsilon = \mathrm{sign}(\kappa_{40}^x) = \mathrm{sign}(\kappa_{04}^x). \tag{22}$$

A rather cumbersome analytical solution (that we call SOK, for "sum of output kurtosis") was given in [3] for the maximization of (22). If the sign of sks ε_γ is substituted for the sign of source kurtosis ε, results of the previous paragraph prove that: a) the SOK solution corresponds to the EML estimator, b) by virtue of this simple modification, I_4 is made applicable, under the EML conditions, to scenarios where the sources may show dissimilar tail (kurtosis sign), and c) the need for the prior knowledge of the source tail is spared, since the sks can be estimated from the available information as in (7).

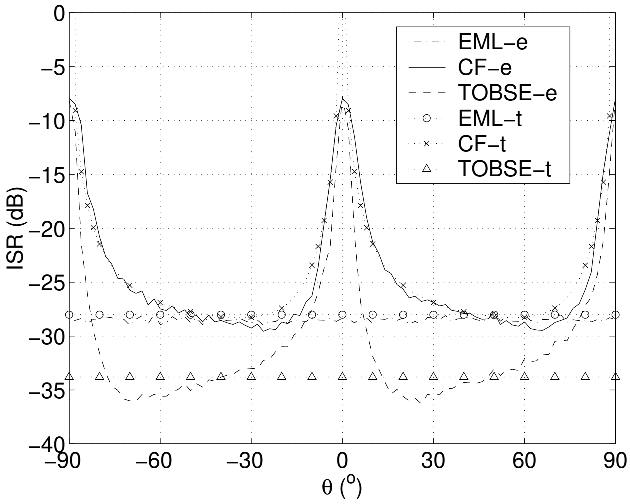

Figure 1: Performance of the EML, CF and TOBSE estimators vs. angle θ, for exponential and asymmetric triangular sources. $T = 5 \cdot 10^3$ samples/signal, 500 Monte Carlo runs. Legend suffixes: 'e', empirical; 't', theoretical.

5. SIMULATIONS

Fig. 1 shows the performance variation of the EML (6), CF (4) and TOBSE (15) estimators, together with their respective expected asymptotic variances, eqns. (9), (10) and (17), as a function of the unknown angular parameter θ. The source pdf's chosen are exponential (long-tailed, asymmetric) and asymmetric triangular (short-tailed), with $T = 5 \cdot 10^3$ samples per signal. The interference-to-signal ratio (ISR) [12] acts as a performance parameter, on the account that it approximates the estimated-angle MSE for values near valid separation solutions [13]. Results are averaged over 500 independent Monte Carlo (MC) iterations. CF performance follows very accurately the theoretically anticipated asymmetric trend, with a severe deterioration around $\theta = k\pi/2$ rad, $k \in \mathbb{Z}$ (Section 2). The EML, which shows a flat response with θ, improves the CF across the whole angle range besides very small regions around -30° and 60°.

TOBSE also exhibits an asymmetric variation with θ for these source distributions. A performance degradation similar to that of the CF is also observed near the 'critical' regions $\theta = k\pi/2$ rad, $k \in \mathbb{Z}$. As a result, the fitness of its analytic variance is precise only outside the cited areas. Further investigations are required to overcome these unforeseen practical limitations, which are probably introduced by the term γ_3' [eqn. (14)] in the estimator expression. However, TOBSE improves, by up to 10 dB, the 4th-order methods over most angle range.

Fig. 2 validates the contrast-function results of Section 4.2. Pseudorandom binary sequences (PRBS's) with appropriate symbol probability are used to simulate a range of source kurtosis values [9], fixing $\kappa_{40}^x = -2$. For SOK,[1]

[1] Original MATLAB code for the SOK solution was down-

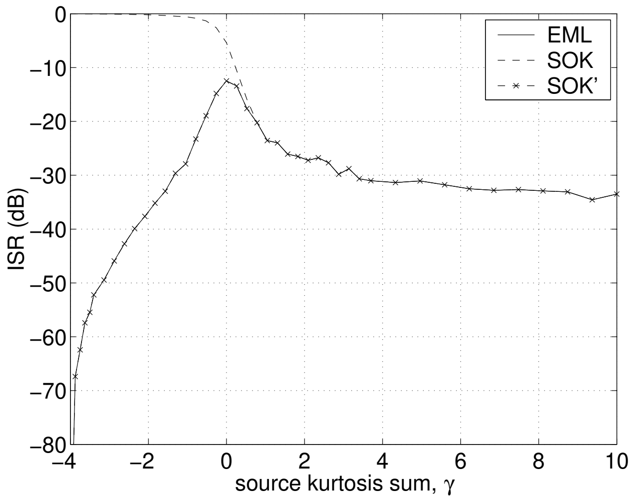

Figure 2: Performance of the EML and SOK estimators against sks, for PRBS sources. $T = 5 \cdot 10^3$ samples/signal, 100 independent Monte Carlo iterations.

$\varepsilon = 1$ is assumed. SOK' corresponds to direct solution SOK, but using ε_γ instead, as estimated from (7). Effectively, SOK' and EML solutions coincide. In fact, estimates from both methods at each MC iteration are identical. SOK and EML estimates are also found to coincide when ε and ε_γ are equal, even though the source tails are then different.

6. CONCLUSIONS AND OUTLOOK

This paper has conceived an innovative unified perception of analytic estimation for BSS in real-mixture scenarios. In the first place, the links among three 4th-order estimators, namely CF, AML and EML, have been elucidated. EML's and CF's asymptotic (large sample) variances have been determined and related. This analysis has shown that the CF performance depends on the unknown parameter to be estimated, with a substantial quality degradation when the angular parameter approaches zero.

In the second place, a general class of estimators based on the data nth-order cumulants has been revealed. At 4th-order, the class yields the EML estimator, whereas a novel 3rd-order estimator (TOBSE) is derived at $n = 3$. TOBSE is valid when there is least one source with asymmetric pdf, and offers potential advantages relative to its 4th-order counterparts due to the lower estimation error of 3rd-order statistics.

Finally, cumulant-matching and contrast-function optimization principles associated with the estimation family have been obtained. At 4th-order, the resulting criterion has allowed the extension of a known contrast function to separation scenarios beyond its original definition.

loaded from `http://www.i3s.unice.fr/~comon/matlab.html`. In order to make all tested methods work from the same cumulant estimates, thus increasing the significance of subsequent comparisons, this original code was modified so that the cumulants were entered as input parameters.

The extension of these unifying results to complex valued mixtures will be presented in forthcoming works.

7. REFERENCES

[1] P. Comon, "Separation of Stochastic Processes", *Proceedings Workshop on Higher-Order Spectral Analysis*, Vail, Colorado, June 1989, pp. 174–179.

[2] P. Comon, "Independent Component Analysis, A New Concept?", *Signal Processing*, Vol. 36, No. 3, April 1994, pp. 287–314.

[3] P. Comon and E. Moreau, "Improved Contrast Dedicated to Blind Separation in Communications", *Proceedings ICASSP'97*, Munich, Germany, April 20–24, 1997, pp. 3453–3456.

[4] F. Harroy and J.-L. Lacoume, "Maximum Likelihood Estimators and Cramer-Rao Bounds in Source Separation", *Signal Processing*, Vol. 55, December 1996, pp. 167–177.

[5] J.-L. Lacoume and P. Ruiz, "Separation of Independent Sources from Correlated Inputs", *IEEE Transactions on Signal Processing*, Vol. 40, No. 12, December 1992, pp. 3074–3078.

[6] J.-L. Lacoume, P.-O. Amblard et P. Comon, *Statistiques d'Ordre Supérieur pour le Traitement du Signal*, Collection Sciences de l'Ingénieur, Paris: Masson, 1997.

[7] E. Moreau and O. Macchi, "High-Order Contrasts for Self-Adaptive Source Separation", *International Journal of Adaptive Control and Signal Processing*, Vol. 10, 1996, pp. 19–46.

[8] A. Stuart and J. K. Ord, *Kendall's Advanced Theory of Statistics*, Vol. I, Sixth Edition, London: Edward Arnold, 1994.

[9] V. Zarzoso and A. K. Nandi. "Modelling Signals of Arbitrary Kurtosis for Testing BSS Methods", *IEE Electronics Letters*, Vol. 34, No. 1, January 8, 1998, pp. 29–30. (Errata: Vol. 34, No. 7, April 2, 1998, p. 703.)

[10] V. Zarzoso and A. K. Nandi, "Blind Source Separation for Virtually Any Source Probability Density Function", *IEEE Transactions on Signal Processing*, Vol. 47, No. 9, September 1999, pp. 2419–2432.

[11] V. Zarzoso and A. K. Nandi, "Blind Source Separation without Optimization Criteria?", *Proceedings ICASSP'99*, Phoenix, Arizona, March 15–19, 1999, Vol. III, pp. 1453–1456.

[12] V. Zarzoso and A. K. Nandi, "Blind Source Separation", in: A. K. Nandi (Ed.), *Blind Estimation Using Higher-Order Statistics*, Boston: Kluwer Academic Publishers, 1999.

[13] V. Zarzoso and A. K. Nandi, "Analysis and Development of Closed-Form Estimators for Blind Separation of Sources — Part I: Real Mixtures", submitted to *IEEE Transactions on Signal Processing*, June 1999.