BLIND SOURCE SEPARATION USING CLOSED-FORM ESTIMATORS WITH OPTIMAL FINITE-SAMPLE PERFORMANCE

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Abstract. We address the blind separation of two source signals from two real-valued instantaneous linear mixtures. Appropriate weighting of previous estimation expressions yields a family of closed-form estimators of the separation parameter which are consistent under rather general conditions. The weight coefficient for the asymptotically (large-sample) most efficient estimator of the family, i.e., the estimator with optimal finite-sample performance, is obtained as a function of the source statistics. Experimental results demonstrate the benefits of the optimal weighted estimator.

1 Introduction

The goal of blind source separation (BSS) [1] is to reconstruct an unknown set of q mutually independent source signals $\mathbf{x} \in \mathbb{R}^q$ which appear mixed at the output of a p-sensor array $\mathbf{y} \in \mathbb{R}^p$, $p \geqslant q$. In the noiseless instantaneous linear case, sources and observations are related via an unknown mixing transformation $M \in \mathbb{R}^{p \times q}$, whose columns contain the source signatures:

$$\mathbf{y} = M\mathbf{x}. \tag{1}$$

The problem consists of estimating the source vector $\mathbf{x}$ and the source-signature matrix M from the exclusive knowledge of sensor vector $\mathbf{y}$. Applications of BSS are found in fields as diverse as radar/sonar, wireless communications, seismic exploration and biomedical signal processing.

Clearly, the source ordering is immaterial in the model above. Also, a scalar factor may be swapped between each source signal and its corresponding signature without altering the sensor output, which renders the source power unidentifiable. To fix this latter indeterminacy, we are free to assume that the power of all sources is unity. When the time structure of the observed processes cannot be exploited (e.g., due to the source spectral whiteness) or is just ignored, one needs to resort to higher-order statistics (HOS). Since a linear mixture of Gaussian random variables is also Gaussian, a necessary condition for the success of HOS-based BSS is that all (except at most one) of the sources be non-Gaussian. A previous second-order spatial whitening process helps to reduce the number of unknowns, resulting in a set of uncorrelated components $\mathbf{z} \in \mathbb{R}^q$ linked to the sources through an unknown orthogonal matrix $Q \in \mathbb{R}^{q \times q}$:

$$\mathbf{z} = Q\mathbf{x}. \tag{2}$$

In the two-signal case, Q becomes an elementary Givens rotation

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{3}$$

and hence the source-signal extraction and mixing-matrix identification reduce to the estimation of angular parameter $\theta \in \mathbb{R}$. The basic two-signal case is of prime importance, since the general scenario $p > 2$ can be tackled through an iterative approach over the signal pairs [2]. Fig. 1 provides a concise pictorial description of the whole mixing-unmixing process.

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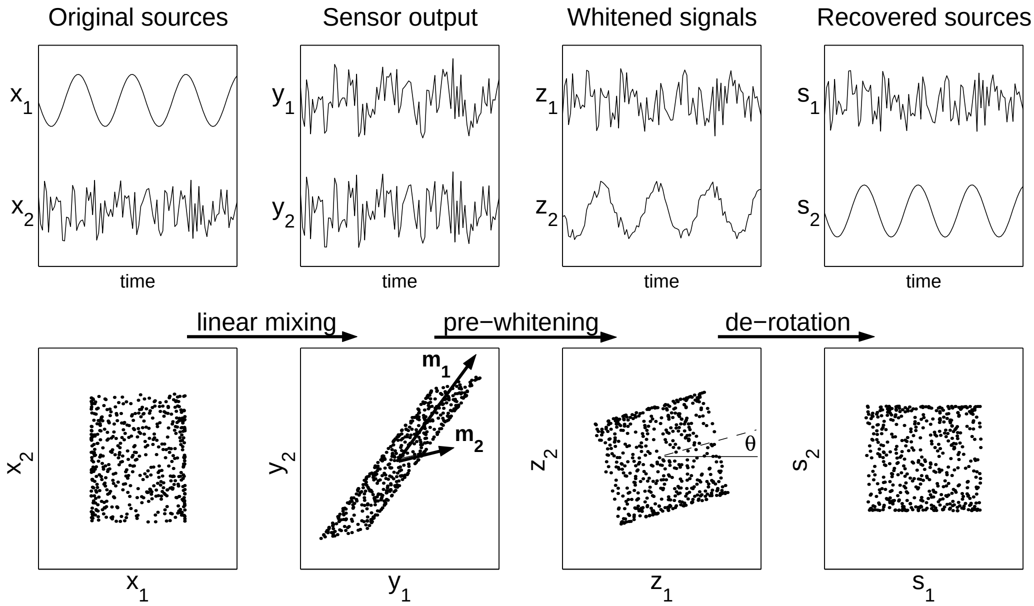

Figure 1: BSS of two instantaneous linear mixtures, with sources composed of a sinusoid and a uniformly distributed process. The top plots show the time variation of the signals. The bottom plots display the corresponding scatter diagrams, which are 2-D representations of the form $\big(x_1(k), x_2(k)\big)$, k denoting a time index. Vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ refer to the source signatures, i.e., the columns of the mixing matrix.

A number of closed-form solutions for the estimation of θ have been developed by different authors. The most attractive feature of closed-form methods is probably the simplicity of their theoretical performance analysis relative to optimization-based procedures. The performance of the first analytic solution [3] was later shown to depend on θ itself [4, 5]. Bogner [6] and Clarke [7] set out the geometric ideas behind the estimation of θ, providing insightful interpretations for source distributions with particular types of symmetry. The maximum-likelihood (ML) approach on the Gram-Charlier expansion of the source probability density function (pdf) produced the solution of [8], whose validity was broadened through the extended ML (EML) and the alternative EML (AEML) estimators [4,9,10]. However, these latter methods lose their consistency under certain source pdf combinations. This deficiency was overcome in [11]. Adopting the ML framework of [8], the two estimators were joined into a single expression, the approximate ML (AML), valid for any source distribution. A similar formula, the MaSSFOC estimator [12], was obtained from the approximate maximization of a contrast function [2] made up of the sum of output squared kurtosis. The notion of linearly combining estimation expressions using arbitrary weights was originally put forward in [11]. It was suggested that the weight parameter could be adjusted by taking advantage of a priori information about the source pdfs, although no specific guidelines were given on how the actual choice should be made.

The ensuing sections study in finer detail this weighted estimator (WE) for BSS and emphasize its potential benefits. The complex-centroid notation used in the EML and AEML estimators allows us, first, to readily establish the connections between the WE and other closed-form solutions and, second and most important, to derive the weight parameter of the asymptotically most efficient WE, i.e., of the estimator with optimal finite-sample performance in the class. The performance gain of the optimal WE relative to other closed-form methods is demonstrated in various computer simulations.

2 Notation

The real-valued whitened observation $\mathbf{z} = (z_1,\, z_2)$ admits a compact formulation in the Argand diagram:

$$\rho \mathrm{e}^{j\phi} = z_1 + j z_2, \qquad j = \sqrt{-1}. \tag{4}$$

This complex-valued representation provides the "z_1 vs. z_2" scatter plot of Fig. 1, in which the x and y axes are substituted by the real and imaginary axes, respectively, of the complex plane. The nth-order cumulants of $\mathbf{z}$ are given by

$$\kappa^z_{n-r,\,r} = \mathrm{Cum}[\underbrace{z_1,\,\ldots,\,z_1}_{n-r},\,\underbrace{z_2,\,\ldots,\,z_2}_{r}]. \tag{5}$$

Symbol $\mu^x_{mn} = \mathrm{E}[x_1^m x_2^n]$ stands for the $(m+n)$th-order moment of the source signals $\mathbf{x} = (x_1,\, x_2)$. Note that the statistical independence of the source signals imply that all their cross-cumulants vanish. We also refer to $\gamma = (\kappa^x_{40} + \kappa^x_{04})$ and $\eta = (\kappa^x_{40} - \kappa^x_{04})$ as the source kurtosis sum (sks) and source kurtosis difference (skd), respectively. Finally, symbol $\mathrm{E}[\cdot]$ denotes the mathematical expectation, and $\angle a$ the principal value of the argument of $a \in \mathbb{C}$.

3 A fourth-order weighted estimator

The complex-centroid based notation developed in [9, 4, 5] proves very convenient in the study of closed-form estimators for BSS. The central idea behind analytic methods of this kind is the search for complex linear combinations of the whitened-sensor statistics which lead to explicit estimation expressions for the parameter of interest. Making use of relationships (2)–(4), it turns out that the 4th-order centroid

$$\xi_4 = \mathrm{E}[\rho^4 \mathrm{e}^{j4\phi}] = (\kappa^z_{40} + \kappa^z_{04} - 6\kappa^z_{22}) + j4(\kappa^z_{31} - \kappa^z_{13}), \tag{6}$$

is equal to $\gamma \mathrm{e}^{j4\theta}$. Therefore, if $\hat{\gamma}$ and $\hat{\xi}_4$ represent sample estimates of γ and ξ_4, respectively, θ may be obtained through

$$\hat{\theta}_{\mathrm{EML}} = \tfrac{1}{4}\angle(\hat{\gamma}\hat{\xi}_4), \tag{7}$$

which is called the EML estimator [4, 9]. The sks can be computed from the whitened array output — thus making the estimator fully blind — through $\gamma = \mathrm{E}[\rho^4] - 8 = \kappa^z_{40} + \kappa^z_{04} + 2\kappa^z_{22}$. Similarly, departing from

$$\xi_2 = \mathrm{E}[\rho^4 \mathrm{e}^{j2\phi}] = (\kappa^z_{40} - \kappa^z_{04}) + j2(\kappa^z_{31} + \kappa^z_{13}) = \eta \mathrm{e}^{j2\theta} \tag{8}$$

one reaches the AEML estimator [4, 10]:

$$\hat{\theta}_{\mathrm{AEML}} = \tfrac{1}{2}\angle\hat{\xi}_2, \tag{9}$$

where $\hat{\xi}_2$ is a suitable sample estimate of ξ_2.[1] Under mild conditions [9,4], $\hat{\theta}_{\mathrm{EML}}$ and $\hat{\theta}_{\mathrm{AEML}}$ are consistent estimators of θ as long as $\gamma \neq 0$ and $\eta \neq 0$, respectively. To alleviate these estimators' inconsistency at null values of sks and skd, a hybrid estimation scheme is proposed in [4, 10], based on a decision rule to choose between the EML and AEML formulae given a batch of whitened-sensor samples. This procedure is called the combined EML (combEML) estimator.

Estimators (7) and (9) can also be merged into a *single* expression. Effectively, observe that the following centroid combination fulfils:

$$\xi_{\mathrm{WE}} = w\gamma\xi_4 + (1-w)\xi_2^2 = \big(w\gamma^2 + (1-w)\eta^2\big)\mathrm{e}^{j4\theta}, \tag{10}$$

and hence

$$\hat{\theta}_{\mathrm{WE}} = \tfrac{1}{4}\angle\hat{\xi}_{\mathrm{WE}}, \qquad 0 < w < 1, \tag{11}$$

produces consistent estimates of θ for any source kurtosis value. Eqn. (11) corresponds to the weighted AML estimator of [11] written in centroid form. Nonetheless, we adhere to the more general denomination of *weighted estimator (WE)*, since its ML nature becomes unclear when extended to the complex-mixture case [13].

When the weight coefficient w takes the values 0, 1/3, 1/2 and 1, respectively, the WE reduces to the AEML [4,10], AML [11], MaSSFOC [12], and EML [9,4] estimators. Hence, different weights correspond to approximate solutions of different optimization criteria: ML in the case of AML, maximization of sum of output square kurtosis for MaSSFOC, etc.

[1] The estimation of η is not necessary, as it may only introduce a potential $\pm\pi/2$-radian bias in $\hat{\theta}$, which does not affect the source recovery (Section 1).

4 Optimal large-sample performance

Estimator (11) shows the shape of the EML (7) and the AEML (9), and hence it may capitalize on the tools used in their analysis [4]. In particular, the asymptotic (large-sample) variance of the WE can also be obtained in closed form, and reads:

$$\sigma^2_{\hat{\theta}_{\mathrm{WE}}} = \frac{\mathrm{E}\Big\{ \big[w\gamma(x_1^3 x_2 - x_1 x_2^3) + (1-w)\eta(x_1^3 x_2 + x_1 x_2^3)\big]^2 \Big\}}{T\big[w\gamma^2 + (1-w)\eta^2\big]^2}, \tag{12}$$

where T is the number of samples per signal. From this expression it turns out that, if $|\kappa_{40}^x| \neq |\kappa_{04}^x|$, the minimum variance estimator of the WE family is obtained for the weight parameter

$$w_{\mathrm{opt}} = \frac{1}{2} + \frac{\mu_{40}^x \mu_{04}^x \big[(\kappa_{40}^x)^2 - (\kappa_{04}^x)^2\big] + \kappa_{40}^x \kappa_{04}^x (\mu_{60}^x - \mu_{06}^x)}{2\big[(\kappa_{40}^x)^2 \mu_{06}^x - (\kappa_{04}^x)^2 \mu_{60}^x\big]}. \tag{13}$$

Hence, given the source statistics, one can select the WE with optimal asymptotic performance. If w_{opt} does not lie in the relevant interval $[0, 1]$, we are left to choose between $w_{\mathrm{opt}} = 0$ (AEML) and $w_{\mathrm{opt}} = 1$ (EML) according to the best expected performance.

5 Experimental results

Several computer simulations illustrate the benefits of the WE and show the goodness of asymptotic variance (12). First, observe that any estimate of the form $\hat{\theta} = \theta + n\pi/2,\ \ n \in \mathbb{Z}$, provides a valid separation solution up to the basic indeterminacies of BSS (Section 1). The interference-to-signal ratio (ISR) [1] is the average relative power contribution from the unwanted original sources in each recovered source. In our particular separation set-up, this performance index can be defined as:

$$\mathrm{ISR} = \mathop{\mathrm{E}}_{i \neq j} \left\{ \frac{\big|(\widehat{Q}^{\mathrm{T}} Q)_{ij}\big|^2}{\big|(\widehat{Q}^{\mathrm{T}} Q)_{ii}\big|^2} \right\}, \tag{14}$$

$\widehat{Q}$ representing the estimated rotation of angle $\hat{\theta}$, and $(A)_{ij}$ the element (i, j) of matrix A (after rearranging its columns to place the dominant entries in its diagonal). The ISR is an objective measure of separation quality, since it is detached from any particular BSS method being used.[2] Also, it approximates the variance of $\hat{\theta}$ around any valid separation solution [4].

Fig. 2 shows the ISR results obtained by the EML, AEML, AML, MaSSFOC and optimal WE, together with the expected asymptotic variances, for varying sample size and i.i.d. sources with uniform and Rayleigh distributions [$w_{\mathrm{opt}} = 0.7141$, from eqn. (13)]. Centroids are computed from their polar forms. The optimal WE substantially outperforms the other estimators, being, e.g., five and ten times as efficient [14] as the AML and the AEML, respectively. The fitness of asymptotic approximation (12) is very accurate in all cases.

The generalized Gaussian distribution (GGD) with shape parameter α, $p(x) \propto \exp(-|x|^\alpha)$, is used as source pdf in the simulation of Fig. 3. We fix $\kappa_{04}^x = 0.5$ and smoothly vary κ_{40}^x to generate a range of sks and skd values. The optimal WE, with w_{opt} calculated as in Section 4 and shown in Fig. 4, is compared with other analytic solutions, a contrast-based approach such as JADE [15], and the Cramér-Rao lower bound (CRLB) obtained in [11] for the real case. The performances of the optimal WE and JADE become identical for sources with equal sign of kurtosis, and follow quite closely the CRLB trend.

[2]The topic of fairness in the choice of a separation performance measure is discussed in [7].

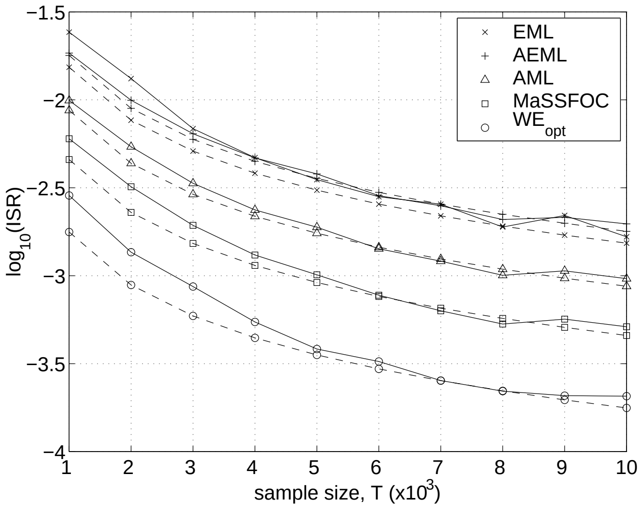

Figure 2: ISR vs. sample size. Uniform–Rayleigh sources, $\theta = 15^{\rm o}$, ν independent Monte Carlo runs, with $\nu T = 5 \times 10^6$. Solid lines: average empirical values. Dashed lines: asymptotic variances (12).

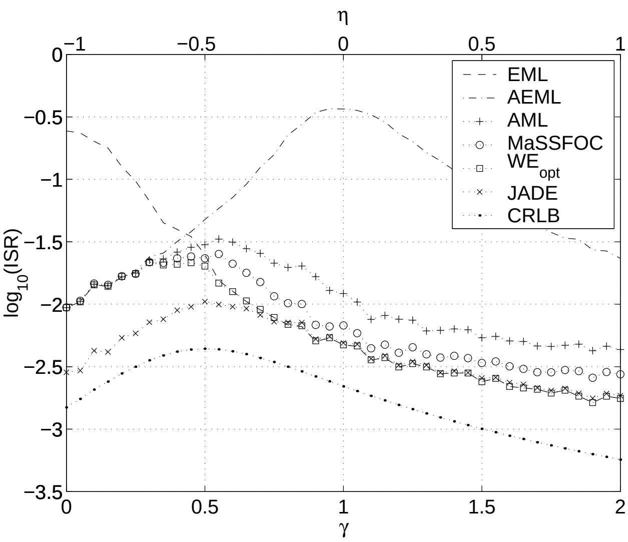

Figure 3: ISR vs. sks γ and skd η. GGD sources, $\kappa_{04}^x = 0.5$, $\theta = 15^{\rm o}$, $T = 5 \times 10^3$ samples, 10^3 Monte Carlo iterations.

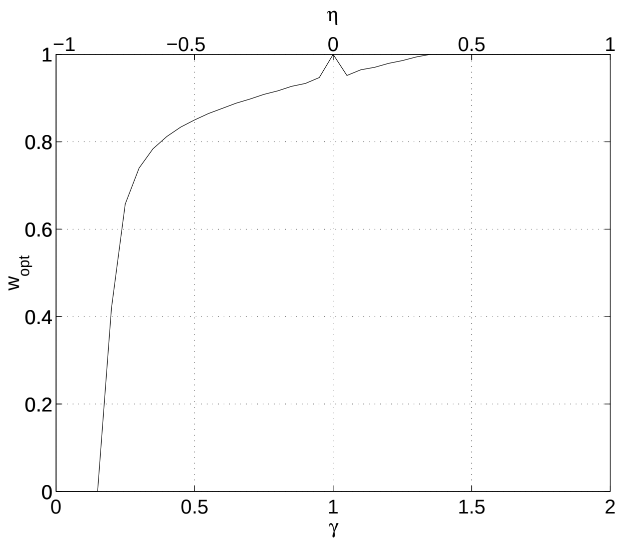

Figure 4: Optimal value of the WE weight parameter in the separation scenario of Fig. 3.

6 Conclusions

HOS-based closed-form solutions to the fundamental real-valued two-signal instantaneous linear mixture BSS problem have been considered. Linear combination of existing fourth-order centroids yields consistent estimates of the separation parameter for any source distribution (provided at most one source is Gaussian). Prior knowledge on the source statistics can be exploited by selecting the estimator with optimal large-sample performance (minimum asymptotic variance).

In order to enable a fully blind operation, it is necessary to develop the optimal weight coefficient as a function of the array-output statistics. The estimators' behaviour in the presence of additive noise and impulsive interference needs to be explored as well. The extension of some of these results to complex mixtures, which are relevant in areas as important as digital communications, is accomplished in [13].

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