CLOSED-FORM SEMI-BLIND SEPARATION OF THREE SOURCES FROM THREE REAL-VALUED INSTANTANEOUS LINEAR MIXTURES VIA QUATERNIONS

Vicente Zarzoso and Asoke K. Nandi

Signal Processing and Communications Group, Department of Electrical Engineering and Electronics,
The University of Liverpool, Brownlow Hill, Liverpool L69 3GJ, UK
Tel/Fax: +44 151 794 4525/4540, `e-mail: {vicente, aknandi}@liv.ac.uk`
`http://www.liv.ac.uk/{˜vicente, EEE/staff/nandi.htm}`

ABSTRACT

In the problem of blind source separation from instantaneous linear mixtures, a unitary transformation remains unknown after second-order spatial whitening. We present a novel approach for the identification of the orthogonal matrix in the real-valued three-signal scenario which, in contrast to existing procedures, operates in a single closed-form step, with no iterations. The new approach is based on intuitive geometrical notions and the theory of quaternions, and develops into a practical semi-blind method requiring certain prior knowledge on the source statistics. A simple numerical experiment illustrates the proposed algorithm.

1. INTRODUCTION

Consider the linear model:

$$\mathbf{y} = M\mathbf{x}, \tag{1}$$

where $\mathbf{y} \in \mathbb{R}^p$, $\mathbf{x} \in \mathbb{R}^q$ and $M \in \mathbb{R}^{p\times q}$. Blind source separation (BSS) aims to recover the unknown source signals $\mathbf{x}$ and mixing matrix M from the observed mixture $\mathbf{y}$ [1]. The above model holds, for instance, when unknown transmitted radio signals impinge on an antenna array whose layout is unknown or difficult to model. The BSS problem is also encountered in a variety of areas such as multi-user communications, radar/sonar, biomedical signal processing and seismic exploration.

The crucial assumption allowing the source extraction and mixing-matrix identification is the statistical independence of the source signals. Mathematically, this assumption can be formulated in terms of the source joint probability density function (pdf) $p_{\mathbf{x}}(\mathbf{x})$:

$$p_{\mathbf{x}}(\mathbf{x}) = \prod_{s=1}^{q} p_{x_s}(x_s), \tag{2}$$

Vicente Zarzoso would like to thank the Royal Academy of Engineering for supporting this work through the award of a Post-doctoral Research Fellowship.

where p_{x_s} is the marginal pdf of the sth component of $\mathbf{x}$. From this perspective, BSS can be accomplished through the independent component analysis (ICA) of the observations. ICA searches for a transformation on the observed vector yielding independent components or, at least, as independent as possible in the sense of the optimization of a suitable independence criterion [2]. Certain identifiability conditions guarantee that the vector obtained via ICA corresponds to the sources, up to, perhaps, irrelevant permutation and scale factors affecting its components. The evident complexity in operating directly over the pdf is alleviated by means of more tractable approximations, or contrasts, based on higher-order statistics [2, 3].

In this paper, we aim to achieve ICA by adopting a more intuitive geometrical viewpoint. After diagonalization of the observed covariance matrix (pre-whitening) — carried out through conventional second-order techniques (principal component analysis)— the mixing reduces to an unknown orthogonal transformation $Q \in \mathbb{R}^{q\times q}$, which can be considered as a rotation in a q-dimensional space. The resulting whitened sensor-output $\mathbf{z} \in \mathbb{R}^q$ then reads:

$$\mathbf{z} = Q\mathbf{x}. \tag{3}$$

Accordingly, $p_{\mathbf{z}}(\mathbf{z}) = p_{\mathbf{x}}(Q^\dagger \mathbf{z})$, where symbol † stands for the transpose operator, so that the pdf of $\mathbf{x}$ undergoes an analogous transformation in the whitened observation signal subspace. In such subspace, the source directions correspond to the columns of Q. The estimated rotation must be such that, when applied on the whitened observations, it aligns the source directions with the observation frame of reference, thus resulting in the pdf of a signal vector with independent components [eqn. (2)].

In the fundamental two-signal case ($q = 2$) the above geometrical concepts are illustrated in Fig. 1. The bottom plots display the scatter diagrams —representations of the form $\big(x_1(\tau), x_2(\tau)\big)$, τ denoting a time index— which are sample approximations of the true pdfs. The unknown unitary transformation reduces to a planar rotation of angle θ, whose estimation can be carried out in closed form [2, 3,

In *Proc. 6th International Symposium on Signal Processing and its Applications*, Kuala Lumpur, Malaysia, August 13-16, 2001, Vol. I, pp. 1-4.

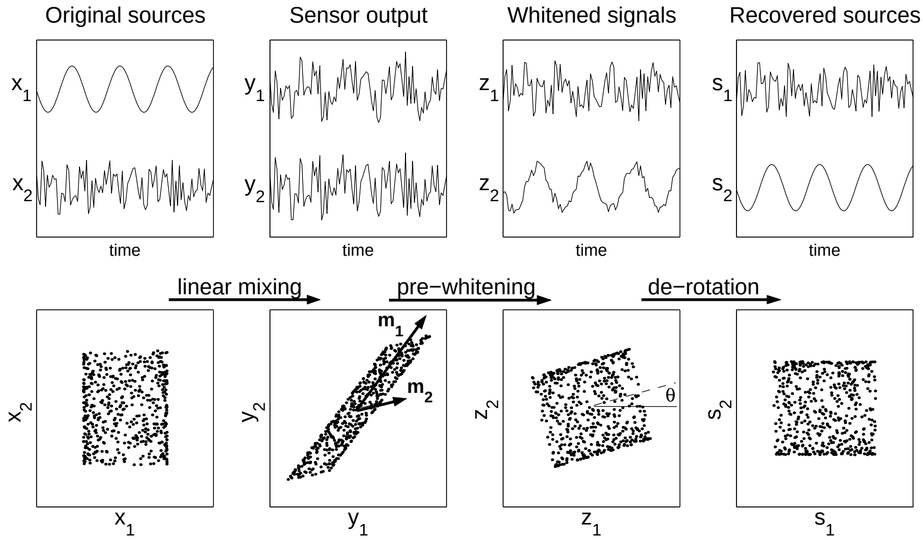

Figure 1: BSS of two instantaneous linear mixtures, with sources composed of a sinusoid and a uniformly distributed process. The top plots show the time variation of the signals, whereas the bottom plots represents the respective scatter plots. Vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ refer to the source directions (or signatures) in the observation signal subspace. After diagonalizing the sensor-output covariance matrix, the source directions simply become the whitened signal subspace basis vectors rotated by an unknown angle θ.

4, 5, 6, 7, 8]. Fourth-order cumulants are directly employed in the algebraic and contrast-based approaches of [4] and [2, 6]. When the sources present symmetric marginal pdfs, a simple yet insightful geometrical standpoint can be taken by exploiting the various symmetries of the resulting scatter plot [5]. The restricted approximate ML criterion of [3] is extended in [7, 8], where the scatter-plot samples are conveniently expressed as complex numbers $(z_1 + iz_2)$, $i^2 = -1$. Higher-order expectations of these representations are shown to generate explicit expressions for the estimation of the relevant parameter.

More than two sources can be separated through the iterative application of a two-signal method over all signal pairs [2]. In the three-signal case ($q = 3$), iterations on three signal pairs are required, sometimes over several sweeps. In a bid to obtain a more efficient separation scheme, the present contribution is devoted to extending the single-step (i.e., non-iterative) closed-form estimation of Q to the three-signal scenario. The mathematical tool which allows us to accomplish this task is the quaternion algebra.

2. QUATERNIONS

Quaternions were invented by Sir William Rowan Hamilton, the most important Irish mathematician of all time, in the 1840's [9]. In his original motivation, Hamilton developed quaternions as quotients of three-dimensional (3D) vectors. Algebraically, a quaternion is a four-dimensional entity that can be represented as a linear combination of the four quaternion units 1, i, j, and k: $\mathbb{A} = a + a_1 i + a_2 j + a_3 k$, $a,\, a_m \in \mathrm{I\!R}$, $1 < m < 3$. These units form the basis for the quaternion space, and fulfil the famous fundamental relations $i^2 = j^2 = k^2 = ijk = -1$, which give the basic rules for quaternion multiplication. Quaternions are the natural extension of complex numbers, with the remarkable feature that their product is not commutative [9]. In fact, they constituted the first non-Abelian ring to be discovered [10]. The most salient properties of quaternions are summarized below [11]:

(P1) Quaternion $\mathbb{A}$ can be expressed as the combination of a scalar part, $a \in \mathrm{I\!R}$, and a vector part, $\mathbf{a} = [a_1,\, a_2,\, a_3]^\dagger \in \mathrm{I\!R}^3$: $\mathbb{A} = [\![a,\, \mathbf{a}]\!]$. We denote $\mathrm{vec}(\mathbb{A}) = \mathbf{a}$.
(P2) Conjugate: $\mathbb{A}^* = [\![a,\, -\mathbf{a}]\!]$.
(P3) Norm: $|\mathbb{A}| = (\mathbb{A}\mathbb{A}^*)^{\frac{1}{2}} = (\mathbb{A}^*\mathbb{A})^{\frac{1}{2}} = \sqrt{a^2 + |\mathbf{a}|^2}$.
(P4) Inverse: $\mathbb{A}\mathbb{A}^{-1} = \mathbb{A}^{-1}\mathbb{A} = 1 \;\Rightarrow\; \mathbb{A}^{-1} = \mathbb{A}^*|\mathbb{A}|^{-2}$.
(P5) Product:

$$\mathbb{A}\mathbb{B} = [\![ab - \mathbf{a}\cdot\mathbf{b},\, a\mathbf{b} + b\mathbf{a} + \mathbf{a}\times\mathbf{b}]\!], \qquad (4)$$

where $\mathbb{B} = [\![b,\, \mathbf{b}]\!]$, symbol '$\cdot$' represents the inner (or dot) product and '$\times$' the vector (or cross) product. Quaternion product is associative, $(\mathbb{A}\mathbb{B})\mathbb{C} = \mathbb{A}(\mathbb{B}\mathbb{C})$, but not commutative, i.e., in general $\mathbb{A}\mathbb{B} \neq \mathbb{B}\mathbb{A}$. Also: $(\mathbb{A}\mathbb{B})^* = \mathbb{B}^*\mathbb{A}^*$.
(P6) Exponential form: $\mathbb{A} = |\mathbb{A}|\mathrm{e}^{\mathbf{n}\phi}$, where

$$\mathrm{e}^{\mathbf{n}\phi} = [\![\cos\phi,\, \mathbf{n}\sin\phi]\!], \qquad (5)$$

and $\mathbf{n} = \mathbf{a}/|\mathbf{a}|$. In addition: $(\mathrm{e}^{\mathbf{n}\phi})^* = \mathrm{e}^{-\mathbf{n}\phi}$, and $\mathrm{e}^{\mathbf{n}\phi}\mathrm{e}^{\mathbf{n}\theta} = \mathrm{e}^{\mathbf{n}\theta}\mathrm{e}^{\mathbf{n}\phi} = \mathrm{e}^{\mathbf{n}(\phi+\theta)}$, $\forall\phi, \theta \in \mathrm{I\!R}$.

One of the most attractive features of quaternions is their ability to represent and perform operations in the 3D space, including affine transformations, projections and, specially, rotations. A point in a 3D Euclidean space, $\mathbf{x} \in \mathbb{R}^3$, can be represented by the pure quaternion $\mathbb{X} = [\![0,\, \mathbf{x}]\!]$. A rotation of angle θ around an axis —or *pole*— $\mathbf{n}$ generates vector $\mathbf{z}$. This rotated point is found in the vector part of another pure quaternion $\mathbb{Z}$ given by the so-called canonical transformation [11]:

$$\mathbb{Z} = \mathrm{e}^{\mathbf{n}\theta/2}\mathbb{X}\,\mathrm{e}^{-\mathbf{n}\theta/2}. \tag{6}$$

Applications of quaternions include molecular and nuclear physics, cryptography, image processing [12], robotics and computer vision [13], computer theory, electromagnetism, and mechanical design. For the first time, this contribution applies quaternions to the problem of source separation.

3. SOURCE SEPARATION VIA QUATERNIONS

3.1. General Approach

The connection between BSS in the three-signal case and quaternions soon becomes apparent. The 3D source and whitened vectors can be represented by quaternions $\mathbb{X}$ and $\mathbb{Z}$, respectively. The unitary transformation Q linking the sources and sensor-output after pre-whitening [eqn. (3)] can similarly be characterized by a pole $\mathbf{n}$ and a rotation angle θ that are both unknown. The quaternion formulation of this transformation is then given by (6). Hence, the problem reduces to the estimation of rotation parameters $(\mathbf{n},\, \theta)$ from $\mathbb{Z}$. If the sources were known, the problem could be solved by the algorithm described next.

Algorithm 1 (Identification of 3D rotation parameters). Given two linearly independent source samples $\mathbf{x}_1$, $\mathbf{x}_2$, and their respective whitened observations, $\mathbf{z}_1$, $\mathbf{z}_2$, the rotation parameters can be identified as follows:

- Step 1. Compute the displacement vectors

$$\mathbf{d}_m = \mathbf{z}_m - \mathbf{x}_m, \qquad m = 1,\, 2. \tag{7}$$

- Step 2. Estimate the rotation axis (Appendix 6.1):
  - If $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{0}$ then $\theta = 0$ and, since there is no rotation, the actual value of $\mathbf{n}$ is irrelevant. Second-order analysis has already performed the source separation.
  - Else, if $\mathbf{d}_1 = \mathbf{0}$, $\mathbf{d}_2 \neq \mathbf{0}$ (resp. $\mathbf{d}_1 \neq \mathbf{0}$, $\mathbf{d}_2 = \mathbf{0}$) then $\mathbf{n} = \mathbf{x}_1$ (resp. $\mathbf{n} = \mathbf{x}_2$).
  - Else, $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$. If $\mathbf{n} = \mathbf{0}$ then $\mathbf{n} = \mathbf{x}_1 \times \mathbf{x}_2 \times \mathbf{d}_m$ ($m = 1$ or $m = 2$).
- Step 3. Normalize pole: $\mathbf{n} := \mathbf{n}/|\mathbf{n}|$.
- Step 4. Set up $\mathbb{N} = [\![0,\, \mathbf{n}]\!]$, $\mathbb{X}_m = [\![0,\, \mathbf{x}_m]\!]$, and $\mathbb{Z}_m = [\![0,\, \mathbf{z}_m]\!]$, $m = 1,\, 2$. Obtain the rotation quaternion as (Appendix 6.2):

$$\mathrm{e}^{\mathbf{n}\theta/2} = \left[(\mathbb{N}\mathbb{Z}_m - \mathbb{Z}_m\mathbb{N})(\mathbb{N}\mathbb{X}_m - \mathbb{X}_m\mathbb{N})^{-1}\right]^{\frac{1}{2}}, \tag{8}$$

  with $m = 1$ or $m = 2$ according to Step 2.

3.2. A Practical Semi-Blind Method

Since, by definition, the source signals are not available, the above procedure cannot be applied directly. Instead, we adopt a *semi-blind* approach, by assuming that we have prior knowledge of the source statistics at least at two different orders. The conditions that these statistics must fulfil will be determined later. First, let us define the r*th-order quaternion moment* [12] as:

$$\overline{\mathbb{X}}_r = \mathrm{E}[\underbrace{\mathbb{X}\mathbb{X}^*\cdots}_{r}], \tag{9}$$

where $\mathrm{E}[\cdot]$ represents the mathematical expectation. Denoting the $(r+s+t)$th-order moment of the source signals as $\mu^x_{rst} = \mathrm{E}[x_1^r x_2^s x_3^t]$, and assuming zero-mean unit-power sources, the first source quaternion moments are:

$$\overline{\mathbb{X}}_1 = 0, \qquad \overline{\mathbb{X}}_2 = 3 \tag{10a}$$

$$\overline{\mathbb{X}}_3 = \mu^x_{300}i + \mu^x_{030}j + \mu^x_{003}k \tag{10b}$$

$$\overline{\mathbb{X}}_4 = \mu^x_{400} + \mu^x_{040} + \mu^x_{004} + 6 \tag{10c}$$

$$\overline{\mathbb{X}}_5 = (\mu^x_{500} + 4\mu^x_{300})i + (\mu^x_{050} + 4\mu^x_{030})j + (\mu^x_{005} + 4\mu^x_{003})k \tag{10d}$$

From the basic properties of quaternions outlined in Section 2, the whitened-signal quaternion moments turn out to be [cf. eqn. (6)]:

$$\overline{\mathbb{Z}}_r = \mathrm{e}^{\mathbf{n}\theta/2}\overline{\mathbb{X}}_r\mathrm{e}^{-\mathbf{n}\theta/2}, \qquad \forall r \geqslant 1. \tag{11}$$

That is, the source quaternion moments are affected, at any order, by the same rotation as the quaternion samples. If we select two orders r_1 and r_2 such that $\overline{\mathbb{X}}_{r_1}$ and $\overline{\mathbb{X}}_{r_2}$ are not proportional, the corresponding moment vectors $\overline{\mathbf{x}}_{r_m} = \mathrm{vec}(\overline{\mathbb{X}}_{r_m})$, $m = 1,\, 2$, are linearly independent. As a conclusion, the rotation parameters can be identified by appropriate substitution of $\overline{\mathbb{X}}_{r_m}$, $\overline{\mathbb{Z}}_{r_m}$, $\overline{\mathbf{x}}_{r_m}$ and $\overline{\mathbf{z}}_{r_m} = \mathrm{vec}(\overline{\mathbb{Z}}_{r_m})$ for $\mathbb{X}_m$, $\mathbb{Z}_m$, $\mathbf{x}_m$ and $\mathbf{z}_m$, $m = 1,\, 2$, resp., in Algorithm 1.

3.3. Identifiability

The identifiability condition of the proposed method reduces to the linear independence of source quaternion moments $\overline{\mathbb{X}}_{r_1}$ and $\overline{\mathbb{X}}_{r_2}$. For this condition to be fulfilled at orders $r_1 = 3$ and $r_2 = 5$, for instance, at least a pair of asymmetrically distributed sources must show dissimilar 3rd- to 5th-order moment ratios. In particular, at most one symmetric distribution is allowed among the sources.

4. ILLUSTRATIVE RESULTS

As an illustrative numerical example, we select $r_1 = 3$, $r_2 = 5$, and source signals composed of 5×10^3 i.i.d. samples with exponential, Rayleigh and uniform distribution. Hence, $\overline{\mathbb{X}}_3 = 2i + 0.63j$ and $\overline{\mathbb{X}}_5 = 52i + 8.52j$ [eqns. (10)], which comply with the conditions set out in Section 3.3. The pole and angle of rotation are $\mathbf{n} = [0.21,\ -0.52,\ 0.83]$ (vector $[2,\ -5, 8]$ normalized) and $\theta = 30^o$, which correspond to an orthogonal mixing matrix $Q = \begin{bmatrix} 0.87 & -0.43 & -0.24 \\ 0.40 & 0.90 & -0.16 \\ 0.28 & 0.05 & 0.96 \end{bmatrix}$. The application of the proposed algorithm on the resulting unitary mixtures produces the estimates $\hat{\mathbf{n}} = [0.03 \pm 0.32,\ -0.54 \pm 0.23,\ 0.72 \pm 0.17]$ and $\hat{\theta} = 36.1^o \pm 17.8^o$, where the "mean ± standard deviation" values are obtained by averaging over 10^3 independent Monte Carlo runs. The interference-to-signal ratio (ISR) [1], a performance index that measures the distance between the original and the estimated mixing matrices, yields an average of $\mathrm{ISR}(Q, \widehat{Q}) = -14.7$ dB, corresponding to a successful source separation.

5. CONCLUSIONS AND OUTLOOK

We have presented a novel approach for three-dimensional linear ICA which enables the closed-form identification of the remaining orthogonal transformation after second-order analysis in a single step, i.e., without iterations of any kind. The approach is based on the algebra of quaternions, and is able to perform non-iterative semi-blind separation of three source signals from three instantaneous linear mixtures.

At the orders considered ($r_1 = 3$, $r_2 = 5$) the applicability conditions of the suggested algorithm are indeed restrictive. Additional work is required to increase the range of source distributions that can be treated. Nevertheless, the basic foundations for the use of quaternions in ICA/BSS have been laid down, and we envisage that the applicability domain of quaternion theory in this exciting signal processing problem will be broadened in future investigations. Further efforts could begin by focusing on the performance analysis of the proposed identification scheme, its comparison with iterative procedures, and the application of quaternion algebra to contrast-based approaches.

6. APPENDICES

6.1. Rotation Identification

We prove that the rotation axis of Q can be identified from two linearly independent source samples $\mathbf{x}_1$, $\mathbf{x}_2$, and their associated observations $\mathbf{z}_1$, $\mathbf{z}_2$, as in Algorithm 1. First, consider the following remarks:

(R1) The eigenspace of rotation $Q \neq I$ (I being the identity matrix) is spanned by its pole $\mathbf{n}$, with associated eigenvalue $\lambda = 1$.

(R2) From $\mathbf{d}_m = (Q - I)\mathbf{x}_m$ and the linear independence of $\mathbf{x}_m$, it follows that $\mathbf{d}_m = \mathbf{0}, \forall m$, iff $Q = I$.

(R3) If $\mathbf{d}_m = \mathbf{0}$ then $\mathbf{x}_m$ belongs to the eigenspace of $Q \neq I$.

(R4) When $\mathbf{d}_m \neq \mathbf{0}$ are parallel, vectors $\mathbf{n}$, $\mathbf{x}_1$ and $\mathbf{x}_2$ are coplanar.

(R5) If $Q \neq I$, the rotation pole lies in the plane perpendicular to any non-null displacement vector $\mathbf{d}_m$.

Therefore:

• If $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{0}$ then, according to (R2), there is no rotation to be identified: $\theta = 0$.

• Else, if $\mathbf{d}_1 = \mathbf{0}$, $\mathbf{d}_2 \neq \mathbf{0}$ (resp. $\mathbf{d}_1 \neq \mathbf{0}$, $\mathbf{d}_2 = \mathbf{0}$) then, from (R1)–(R3), the rotation axis is spanned by $\mathbf{x}_1$ (resp. $\mathbf{x}_2$).

• Else, $\mathbf{d}_1 \times \mathbf{d}_2 = 0$ implies that $\mathbf{d}_m$ are parallel and hence, from (R4)–(R5), $\mathbf{n}$ can be computed from the intersection of the plane spanned by vectors $\mathbf{x}_m$ and the plane perpendicular to either $\mathbf{d}_m$. If $\mathbf{d}_m$ are not parallel, (R5) guarantees that $\mathbf{n}$ can be obtained from their vector product.

6.2. Rotation Quaternion

It is shown next that the quaternion associated with a rotation around a pole $\mathbf{n}$ applied to point $\mathbf{x}$ resulting in another point $\mathbf{z}$ is given by

$$\mathrm{e}^{\mathbf{n}\theta/2} = \left[(\mathbb{NZ} - \mathbb{ZN})(\mathbb{NX} - \mathbb{XN})^{-1}\right]^{\frac{1}{2}}, \tag{12}$$

where $\mathbb{N} = [\![0,\ \mathbf{n}]\!]$, $\mathbb{X} = [\![0,\ \mathbf{x}]\!]$ and $\mathbb{Z} = [\![0,\ \mathbf{z}]\!]$.

From quaternion product (4), we have that $\mathbf{u} = \mathrm{vec}(\mathbb{NZ} - \mathbb{ZN}) = 2(\mathbf{n} \times \mathbf{z})$ and, similarly, $\mathbf{v} = \mathrm{vec}(\mathbb{NZ} - \mathbb{ZN}) = 2(\mathbf{n} \times \mathbf{z})$. Now, since $\mathbf{x}$ is rotated around $\mathbf{n}$ by θ radians to yield $\mathbf{z}$, it turns out that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to $\mathbf{n}$, and separated by the same angular distance. Considering the associated pure quaternions $\mathbb{U} = [\![0,\ \mathbf{u}]\!]$, $\mathbb{V} = [\![0,\ \mathbf{v}]\!]$, and from the properties summarized in Section 2: $\mathbb{VU}^{-1} = [\![\mathbf{u} \cdot \mathbf{v},\ \mathbf{u} \times \mathbf{v}]\!] |\mathbf{u}|^{-2} = [\![\cos\theta,\ \mathbf{n}\sin\theta]\!] = \mathrm{e}^{\mathbf{n}\theta}$, from which result (12) readily follows. Finally, observe that, since scalars do commute in the quaternion product, quaternion $(-\mathbb{N})$ also yields $\mathrm{e}^{\mathbf{n}\theta/2}$ in (12). In such a case, the equivalent rotation parameters $(-\mathbf{n},\ -\theta)$ are estimated instead of $(\mathbf{n},\ \theta)$.

7. REFERENCES

[1] V. Zarzoso and A. K. Nandi, "Blind Source Separation," in *Blind Estimation Using Higher-Order Statistics*, A. K. Nandi (Ed.), pp. 167–252. Kluwer Academic Publishers, Boston, 1999.

[2] P. Comon, "Independent Component Analysis, A New Concept?," *Sig. Proc.*, Vol. 36, No. 3, pp. 287–314, Apr. 1994.

[3] F. Harroy and J.-L. Lacoume, "Maximum Likelihood Estimators and Cramer-Rao Bounds in Source Separation," *Sig. Proc.*, Vol. 55, No. 2, pp. 167–177, Dec. 1996.

[4] P. Comon, "Separation of Sources Using Higher-Order Cumulants," in *Proc. SPIE*, San Diego, CA, 1989, Vol. 1152, pp. 170–181.

[5] R. E. Bogner, *Blind Separation of Sources*, Memo. No. 4559, Defence Research Agency, Malvern, UK, May 1992.

[6] F. Herrmann and A. K. Nandi, "Maximisation of Squared Cumulants for Blind Source Separation," *IEE Electronics Letters*, Vol. 36, No. 19, pp. 1164–1165, 2000.

[7] V. Zarzoso and A. K. Nandi, "Blind Separation of Independent Sources for Virtually Any Source Probability Density Function," *IEEE Trans. Sig. Proc*, Vol. 47, No. 9, pp. 2419–2432, Sept. 1999.

[8] V. Zarzoso, *Closed-Form Higher-Order Estimators for Blind Separation of Independent Source Signals in Instantaneous Linear Mixtures*, Ph.D. thesis, The University of Liverpool, UK, Oct. 1999.

[9] W. R. Hamilton, *Elements of Quaternions*, Longmans, Green & Co., London, 1866.

[10] J. A. Gallian, *Contemporary Abstract Algebra*, Houghton Mifflin, Boston, MA, 4th edition, 1998.

[11] S. L. Altmann, *Rotations, Quaternions and Double Groups*, Clarendon Press, Oxford, 1986.

[12] S.-C. Pei and C.-M. Cheng, "Color Image Processing by Using Binary Quaternion-Moment-Preserving Thresholding Technique," *IEEE Transactions on Image Processing*, Vol. 8, No. 5, pp. 614–628, May 1999.

[13] R. Smith, A. Frost, and P. Probert, "Sensor System for the Navigation of an Underwater Vehicle," *International Journal of Robotics Research*, Vol. 18, No. 7, pp. 697–710, 1999.