WEIGHTED CLOSED-FORM ESTIMATORS FOR BLIND SOURCE SEPARATION

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ABSTRACT

This paper investigates a novel closed-form estimation class, so-called weighted estimator (WE), for blind source separation in the basic two-signal problem. Proper combination of previously proposed estimators yields consistent estimates of the separation parameters under general conditions. In the real-mixture case, we determine analytic expressions for the WE asymptotic (large-sample) variance and the source-dependent weight value of the most efficient estimator in the class. By means of the bicomplex-number formalism, the WE is extended to the complex-mixture scenario, for which Cramér-Rao bounds are also derived. Simulations compare the WE with other methods, demonstrating its potential.

Keywords: blind source separation, estimation theory, higher-order statistics, non-Gaussian signal processing, sensor array processing.

1. INTRODUCTION

The problem of blind source separation (BSS) arises in a great variety of applications, in fields as diverse as wireless communications, seismic exploration and biomedical signal processing. BSS aims to reconstruct an unknown set of q mutually independent source signals $\mathbf{x} \in \mathbb{C}^q$ which appear mixed at the output of a p-sensor array $\mathbf{y} \in \mathbb{C}^p$, $p \geqslant q$. In the noiseless instantaneous linear case, sources and observations are linked through an unknown mixing transformation $M \in \mathbb{C}^{p\times q}$:

$$\mathbf{y} = M\mathbf{x}. \quad (1)$$

The problem consists of estimating the source vector $\mathbf{x}$ and the mixing matrix M from the exclusive knowledge of sensor vector $\mathbf{y}$. Neither the ordering nor the power and phase-shift of the sources can be identified in the model above, so we may assume, with no loss of generality, an identity source covariance matrix.

When the time structure of the signals cannot be exploited (e.g., due to the source spectral whiteness), one needs to resort to higher-order statistics (HOS) [1]. The success of the separation then relies on the non-Gaussian nature of the sources. A previous spatial whitening process (entailing second-order decorrelation and power normalization) helps to reduce the number of unknowns, resulting in a set of normalized uncorrelated components $\mathbf{z} \in \mathbb{C}^q$:

$$\mathbf{z} = Q\mathbf{x}, \quad (2)$$

with $Q \in \mathbb{C}^{q\times q}$ unitary. As the general scenario $p > 2$ can be tackled through an iterative approach over the signal pairs [2], the two-signal case, $p = q = 2$, is of fundamental importance. The unitary transformation Q is then a complex elementary Givens rotation matrix:

$$Q = \begin{bmatrix} \cos\theta & -\mathrm{e}^{-j\alpha}\sin\theta \\ \mathrm{e}^{j\alpha}\sin\theta & \cos\theta \end{bmatrix}. \quad (3)$$

Hence, the source-signal extraction and mixing-matrix identification reduce to the estimation of angular parameters $\theta,\ \alpha \in \mathbb{R}$.

In the real-valued mixture case, $\alpha = 0$ and only θ is unknown. The performance of the first closed-form solution for the estimation of θ, based on the output 4th-order cross-cumulant nulling [3], was later shown to depend on θ itself [4, 5]. The maximum-likelihood (ML) approach on the Gram-Charlier expansion of the source probability density function (pdf) produced the solution of [6], whose validity was broadened through the extended ML (EML) and the alternative EML (AEML) estimators [4, 7, 8]. Such estimators lose their consistency for zero source kurtosis sum (sks) and source kurtosis difference (skd), respectively. This deficiency was overcome in [8] and [9]. In the latter, adopting the framework of [6] the two estimators were joined into a single analytic expression, the approximate ML (AML). The MaSSFOC estimator [10], derived from the approximate maximization of a contrast function made up of the sum of output squared kurtosis [2], exhibits a strikingly resembling form. The notion of linearly combining estimation expressions using arbitrary weights was originally put forward in [9], giving rise to the so-called weighted AML (WAML) estimator. It was suggested that the weight parameter could be adjusted by taking advantage of a priori information on the source pdfs, although no specific guidelines were given on how the actual choice should be made.

The present contribution fills this gap by studying in finer detail this weighted estimator (WE) for BSS and emphasizing its potential benefits. In the real-mixture case, we capitalize on the complex-centroid notation used in the EML and AEML estimators in order to provide an analytic formula for the WE large-sample variance. From this formula, the weight parameter of the asymptotically most efficient WE is obtained as a function of the source statistics. In addition, the WE is neatly extended to the complex-valued mixture case with the bicomplex number formalism developed in [4, 11]. We deduce Cramér-Rao lower bounds (CRLBs) for the pertinent parameters, and show in simulations that the WE is able to follow the CRLB trend of an objective separation-quality performance index. The connections between the WE and other analytic solutions are also highlighted throughout the paper.

First, we summarize a few mathematical notations. Symbol $\mu^x_{mn} = \mathrm{E}[x_1^m x_2^n]$, where $\mathrm{E}[\cdot]$ denotes the mathematical expectation, stands for the $(m+n)$th-order moment of the source signals $\mathbf{x} = (x_1,\, x_2)$. For convenience, the cumulants of complex vector $\mathbf{z} = (z_1,\, \ldots,\, z_q)$ are defined as $\mathrm{Cum}^z_{i_1 i_2 i_3\ldots} =$

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$\mathrm{Cum}[z_{i_1}^*, z_{i_2}, z_{i_3}^*, \ldots]$, $1 \leqslant i_k \leqslant q$, with the convention, in the two-component case, $\kappa^z_{n-r,\,r} = \mathrm{Cum}^z_{\underbrace{1\ldots1}_{n-r}\underbrace{2\ldots2}_{r}}$. We also define $\gamma = \kappa^x_{40} + \kappa^x_{04}$ (sks) and $\eta = \kappa^x_{40} - \kappa^x_{04}$ (skd). Symbol $\angle a$ represents the principal value of the argument of $a \in \mathbb{C}$.

2. REAL-MIXTURE CASE

2.1. Fourth-Order Weighted Estimator

The WAML estimator [9] accepts a more convenient formulation when adopting the EML/AEML approach [4, 5, 7, 8], which is based on the polar representation of real-valued bivariate random vector $\mathbf{z} = (z_1, z_2)$ as $\rho \mathrm{e}^{j\phi} = z_1 + jz_2$, $j = \sqrt{-1}$. Higher-order expectations then generate complex-valued linear combinations (*centroids*) of the whitened-sensor statistics which lead to explicit estimation expressions for the parameter of interest. Accordingly, the EML is expressed as

$$\hat{\theta}_{\mathrm{EML}} = \tfrac{1}{4}\angle(\gamma\xi_4), \tag{4}$$

where ξ_4 is the 4th-order complex centroid:

$$\xi_4 = \mathrm{E}[\rho^4 \mathrm{e}^{j4\phi}] = (\kappa^z_{40} + \kappa^z_{04} - 6\kappa^z_{22}) + j4(\kappa^z_{31} - \kappa^z_{13}), \tag{5}$$

and the sks can be estimated from the array output through $\gamma = \mathrm{E}[\rho^4] - 8 = \kappa^z_{40} + \kappa^z_{04} + 2\kappa^z_{22}$. Similarly, the AEML [4, 8] reads:

$$\hat{\theta}_{\mathrm{AEML}} = \tfrac{1}{2}\angle\xi_2, \tag{6}$$

$$\xi_2 = \mathrm{E}[\rho^4 \mathrm{e}^{j2\phi}] = (\kappa^z_{40} - \kappa^z_{04}) + j2(\kappa^z_{31} + \kappa^z_{13}). \tag{7}$$

Under mild conditions [4, 7], centroids ξ_4 and ξ_2 are consistent estimators of $\gamma \mathrm{e}^{j4\theta}$ and $\eta \mathrm{e}^{j2\theta}$, respectively, so that $\hat{\theta}_{\mathrm{EML}}$ and $\hat{\theta}_{\mathrm{AEML}}$ consistently estimate θ as long as $\gamma \neq 0$ and $\eta \neq 0$, respectively. It follows that

$$\hat{\theta}_{\mathrm{WE}} = \tfrac{1}{4}\angle\xi_{\mathrm{WE}}, \qquad \text{with} \tag{8}$$

$$\xi_{\mathrm{WE}} = w\gamma\xi_4 + (1-w)\xi_2^2, \qquad 0 < w < 1. \tag{9}$$

is a consistent estimator of θ for *any* source distribution (besides when the sources are both Gaussian). Eqn. (8) is essentially the WAML estimator [9] written in centroid form. Nonetheless, we adhere to the more general denomination of *weighted estimator (WE)*, since its ML nature becomes unclear when extended to the complex-signal domain (Section 3).

Some special cases of the WE are:

(i) $w = 0$: AEML estimator of [4, 8].
(ii) $w = 1/3$: AML estimator of [9].
(iii) $w = 1/2$: MaSSFOC estimator of [10].
(iv) $w = 1$: EML estimator of [4, 7].

2.2. Performance Analysis

Along the lines of [4, 5], and omitting tedious algebraic details, the asymptotic (large-sample) variance of the WE (8) is determined as:

$$\sigma^2_{\hat{\theta}_{\mathrm{WE}}} = \frac{\mathrm{E}\Big\{\big[w\gamma(x_1^3x_2 - x_1x_2^3) + (1-w)\eta(x_1^3x_2 + x_1x_2^3)\big]^2\Big\}}{T\big[w\gamma^2 + (1-w)\eta^2\big]^2}, \tag{10}$$

where T is the number of samples. Remark that:

(i) $\sigma^2_{\hat{\theta}_{\mathrm{WE}}}$ reduces to the asymptotic variance of the AEML and EML estimators [4, 5] for $w = 0$ and $w = 1$, respectively.

(ii) When $\gamma = 0$ (resp. $\eta = 0$), WE performance reduces to that of the AEML (resp. EML) estimator, for any $0 < w < 1$.

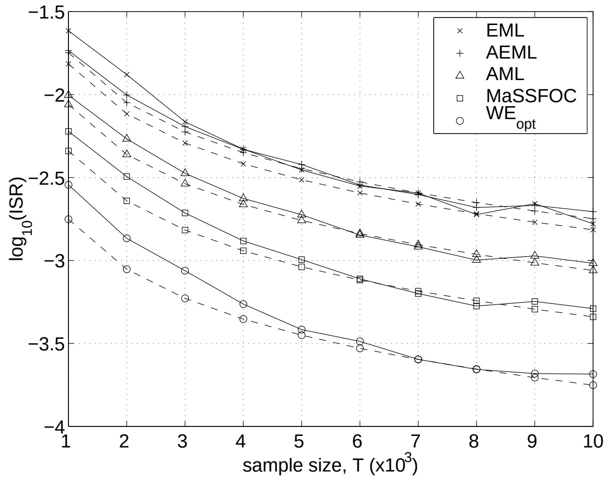

Fig. 1. ISR vs. sample size. Uniform–Rayleigh sources, $\theta = 15^{\mathrm{o}}$, ν independent Monte Carlo runs, with $\nu T = 5 \times 10^6$. Solid lines: average empirical values. Dashed lines: asymptotic variances (10).

2.3. Optimal Large-Sample Performance

If $|\kappa^x_{40}| \neq |\kappa^x_{04}|$, the derivative of eqn. (10) with respect to w cancels at:

$$w_{\mathrm{opt}} = \frac{1}{2} + \frac{\mu^x_{40}\mu^x_{04}\big[(\kappa^x_{40})^2 - (\kappa^x_{04})^2\big] + \kappa^x_{40}\kappa^x_{04}(\mu^x_{60} - \mu^x_{06})}{2\big[(\kappa^x_{40})^2\mu^x_{06} - (\kappa^x_{04})^2\mu^x_{60}\big]}. \tag{11}$$

Since $\partial^2(\sigma^2_{\hat{\theta}_{\mathrm{WE}}})/\partial w^2\big|_{w_{\mathrm{opt}}} > 0$, w_{opt} corresponds to the minimum variance estimator of the WE family. Hence, given the source statistics, one can select the WE with optimal asymptotic performance. If $w_{\mathrm{opt}} \notin [0, 1]$, we choose between $w_{\mathrm{opt}} = 0$ (AEML) and $w_{\mathrm{opt}} = 1$ (EML) the value that gives the lowest $\sigma^2_{\hat{\theta}_{\mathrm{WE}}}$ in (10).

2.4. Simulation Results

A few simulations illustrate the benefits of the WE and show the goodness of asymptotic approximation (10). First, observe that any angle estimate of the form $\hat{\theta} = \theta + n\pi/2$, $n \in \mathbb{Z}$, provides a valid separation solution up to the indeterminacies mentioned in Sec. 1. The interference-to-signal ratio (ISR) performance index [1] approximates the variance of $\hat{\theta}$, $\sigma^2_{\hat{\theta}}$, around any valid separation solution [4]. The ISR is an objective measure of separation performace, for it is method independent.

Fig. 1 shows the ISR results obtained by the EML, AEML, AML, MaSSFOC and optimal WE, together with the expected asymptotic variances, for varying sample size and i.i.d. sources with uniform and Rayleigh distributions [$w_{\mathrm{opt}} = 0.7141$, from eqn. (11)]. Centroids are computed from their polar forms. The optimal WE substantially outperforms the other estimators, being, e.g., five and ten times as efficient [12] as the AML and the AEML, respectively. The fitness of asymptotic approximation (10) is very precise in all cases.

The generalized Gaussian distribution (GGD) with shape parameter λ, $p(x) \propto \exp(-|x|^\lambda)$, is used as source pdf in the simulation of Fig. 2. We fix $\kappa^x_{04} = 0.5$ and smoothly vary κ^x_{40} to generate

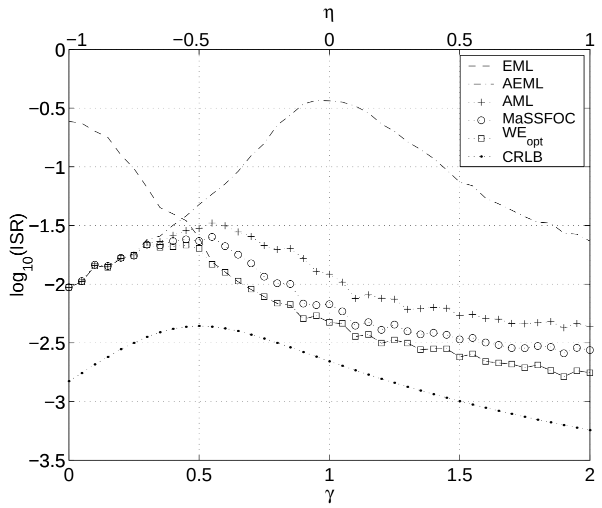

Fig. 2. ISR vs. sks γ and skd η. GGD sources, $\kappa_{04}^x = 0.5$, $\theta = 15^{\mathrm{o}}$, $T = 5 \times 10^3$ samples, 10^3 Monte Carlo runs.

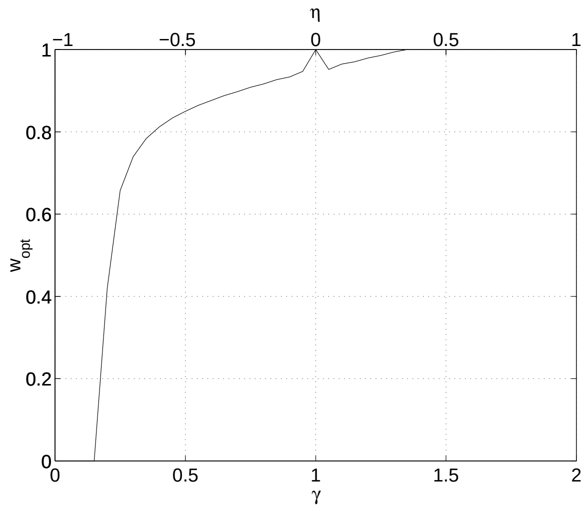

Fig. 3. Optimal value of the WE weight parameter in the separation scenario of Fig. 2.

a range of sks and skd values. The optimal WE, with w_{opt} calculated as in Sec. 2.3 and shown in Fig. 3, is compared with other analytic solutions and the CRLB obtained in [9] for the real case. The optimal WE follows the CRLB more closely than any of the other methods.

3. COMPLEX-MIXTURE CASE

3.1. Bicomplex Numbers

In [4, 11], the so-called bicomplex numbers prove useful in simplifying the development of closed-form estimators in the complex-mixture scenario. Given a unitary matrix $Q = \left[\begin{smallmatrix} a & -b^* \\ b & a^* \end{smallmatrix}\right]$, $a, b \in \mathbb{C}$, where * denotes complex conjugation, the associated bicomplex number is defined as $\bar{x} = a + \jmath b$. Though analogous to j, the *bimaginary unit* $\jmath$ is actually a distinct algebraic element. Terms $a = \mathbb{R}\mathrm{e}(\bar{x})$ and $b = \mathbb{I}\mathrm{m}(\bar{x})$ are the *breal* and *bimaginary* parts of $\bar{x}$, respectively. The product of two bicomplex numbers $\bar{x}_1 = a_1 + \jmath b_1$ and $\bar{x}_2 = a_2 + \jmath b_2$ is defined in accordance with the product of unitary transformations:

$$\bar{x}_1 \bar{x}_2 = (a_1 a_2 - b_1^* b_2) + \jmath (b_1 a_2 + a_1^* b_2). \tag{12}$$

In this manner, an isomorphism is created between the set of unitary matrices under usual matrix product and the set of bicomplex numbers under the above product operation. Note that, as with j, $\jmath^2 = -1$. A special class of bicomplex numbers arises when the associated unitary transformation shows the shape of (3):

$$\mathrm{e}_{\alpha}^{\jmath\theta} = \cos\theta + \jmath \mathrm{e}^{j\alpha} \sin\theta, \tag{13}$$

which we call bicomplex exponential.

3.2. Fourth-Order Weighted Estimator

By means of the bicomplex formalism, one can easily generalize centroids (5) and (7) to the complex-mixture case. Effectively,

$$\bar{\xi}_4 = (\kappa_{40}^z + \kappa_{04}^z - 6\kappa_{22}^z) + \jmath 4(\kappa_{31}^z - \kappa_{13}^z) \tag{14}$$

and

$$\bar{\xi}_2 = (\kappa_{40}^z - \kappa_{04}^z) + \jmath 2(\kappa_{31}^z + \kappa_{13}^z) \tag{15}$$

are consistent estimators of $\gamma \mathrm{e}_{\alpha}^{\jmath 4\theta}$ and $\eta \mathrm{e}_{\alpha}^{\jmath 2\theta}$, respectively, under the same general conditions as in the real case. Centroid (14) gives rise to the complex EML (CEML) estimator [4, 11], whereas (15) yields the complex AEML (CAEML) estimator [4]. Bearing in mind the bicomplex product (12), it follows immediately that the linear combination

$$\bar{\xi}_{\mathrm{CWE}} = w\gamma\bar{\xi}_4 + (1-w)\bar{\xi}_2^2 \tag{16}$$

consistently estimates $\big(w\gamma^2 + (1-w)\eta^2\big)\mathrm{e}_{\alpha}^{\jmath 4\theta}$. The sks γ may be obtained from the available data just as in the real case. For $w \in [0, 1]$, parameters (θ, α) are estimated through

$$\begin{cases} 4\hat{\theta}_{\mathrm{CWE}} = \angle\big(\mathbb{R}\mathrm{e}(\bar{\xi}_{\mathrm{CWE}}) + j|\mathbb{I}\mathrm{m}(\bar{\xi}_{\mathrm{CWE}})|\big) \\ \hat{\alpha}_{\mathrm{CWE}} = \angle\, \mathbb{I}\mathrm{m}(\bar{\xi}_{\mathrm{CWE}}), \end{cases} \tag{17}$$

which is the *complex WE (CWE)*.

3.3. Cramér-Rao Lower Bounds

Assuming circularly distributed source signals composed of T independent samples, the Fisher information matrix (FIM) for the estimation of parameters (θ, α) in model (2)–(3) reads:

$$\mathrm{FIM}_{(\theta,\,\alpha)} = T \begin{bmatrix} I & 0 \\ 0 & \frac{1}{4} I \sin^2 2\theta \end{bmatrix}, \tag{18}$$

where

$$I = I_1 + I_2 - 4,$$
$$I_k = \frac{1}{2} \iint_{\mathsf{D}_k} \frac{1}{p_k} \left[\left(\frac{\partial p_k}{\partial u}\right)^2 + \left(\frac{\partial p_k}{\partial v}\right)^2 \right] \mathrm{d}u\mathrm{d}v, \tag{19}$$

and $p_k(u, v)$ is the pdf of the kth source signal $x_k = u_k + jv_k$, $u_k, v_k \in \mathbb{R}$, $k = 1, 2$. Integration extends over the definition domain D_k of the corresponding random variable.

It is interesting to note that:

(i) The CRLBs of θ and α are decoupled, and therefore:

$$\mathrm{CRLB}_\theta = (TI)^{-1} \tag{20}$$
$$\mathrm{CRLB}_\alpha = 4(TI\sin^2 2\theta)^{-1} \tag{21}$$

(ii) For sources with complex generalized Gaussian distribution (CGGD) of shape parameter λ, given by

$$p(u,\,v) \propto \exp\{-(u^2+v^2)^{\frac{\lambda}{2}}\}, \quad \lambda > 0, \tag{22}$$

we have

$$I_k = \tfrac{1}{2}\lambda_k^2\Gamma(4/\lambda_k)/\Gamma^2(2/\lambda_k). \tag{23}$$

Then, the FIM is zero, and hence the model unidentifiable, iff $\lambda_1 = \lambda_2 = 2$, i.e., both sources are Gaussian.

(iii) When $\theta = n\pi/2$, $\forall n \in \mathbb{Z}$, estimation of α becomes unfeasible. However, in such cases the correct estimation of α does not affect the source extraction, e.g., if $\theta = 0$, Q in (3) is just an identity matrix; if $\theta = \pi/2$, Q only contains off-diagonal phase factors which are 'absorbed' by the source signals.

(iv) Endorsing the previous point we have that, for accurate estimates of $(\theta,\,\alpha)$, ISR $\approx \sigma^2_{\hat\theta} + \frac{1}{4}\sigma^2_{\hat\alpha}\sin^2 2\theta$, so that ISR is lower bounded by $2 \times \mathrm{CRLB}_\theta$. When $\theta = n\pi/2$, $n \in \mathbb{Z}$, and if $\hat\theta$ is still precise enough, this bound decreases to CRLB_θ. That is, the lower bound of separation-performance objective measure ISR is independent of θ and is (asymptotically) determined by the source statistics only [via I in (19)].

3.4. Simulation Results

A simple simulation experiment compares the behaviour of the CEML, CAEML and CWE (with $w = 1/3$ and $w = 1/2$, which would correspond to the complex extensions of AML and MaSS-FOC, resp.). Two independent CGGDs are used as sources. Average ISR results as a function of sks and skd are displayed in Fig. 4. As expected, the CEML and CAEML worsen near $\gamma = 0$ and $\eta = 0$, respectively. By contrast, the CWE maintains a satisfactory separation in both tested cases over all γ and η range, and, as occurred in the real case (Fig. 2), its performance follows closely the CRLB trend.

4. CONCLUSIONS AND OUTLOOK

A new class of closed-form estimators of the separation parameters in the fundamental two-signal instantaneous linear mixture BSS problem has been investigated. A weighted estimator (WE) arises from the linear combination of the EML and AEML centroids, and produces consistent estimates under rather general conditions (essentially, if at most one source is Gaussian). For real-valued mixtures, prior knowledge on the source statistics can be exploited by selecting the WE with optimal large-sample performance (minimum asymptotic variance). With the aid of the bicomplex numbers the WE has also been extended to the complex-mixture case, where it has shown a performance variation similar to the CRLB, that we have derived for circular sources.

Paths of further research include the asymptotic performance analysis of the WE in the complex environment, which is of relevance in areas as important as digital communications. Also, in order to enable a fully blind operation, it is necessary to develop the optimal weight coefficient as a function of the array-output statistics. The estimator's behaviour in the presence of additive noise and impulsive interference needs to be explored as well.

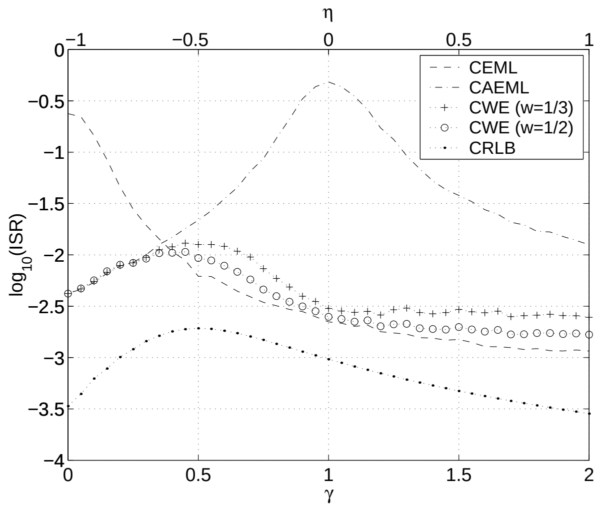

Fig. 4. ISR vs. sks γ and skd η. CGGD sources, $\kappa^x_{04} = 0.5$, $\theta = 15^{\text{o}}$, $\alpha = 65^{\text{o}}$, $T = 5 \times 10^3$ samples, 10^3 independent Monte Carlo iterations.

5. REFERENCES

[1] V. Zarzoso and A. K. Nandi, "Blind Source Separation," in *Blind Estimation Using Higher-Order Statistics*, A. K. Nandi (Ed.), pp. 167–252. Kluwer Academic Publishers, Boston, 1999.

[2] P. Comon, "Independent Component Analysis, A New Concept?," *Signal Processing*, Vol. 36, No. 3, pp. 287–314, Apr. 1994.

[3] P. Comon, "Separation of Sources Using Higher-Order Cumulants," in *Proc. SPIE*, San Diego, CA, 1989, Vol. 1152, pp. 170–181.

[4] V. Zarzoso, *Closed-Form Higher-Order Estimators for Blind Separation of Independent Source Signals in Instantaneous Linear Mixtures*, Ph.D. thesis, The University of Liverpool, UK, Oct. 1999.

[5] V. Zarzoso and A. K. Nandi, "Unified Formulation of Closed-Form Estimators for Blind Source Separation in Real Instantaneous Linear Mixtures," in *Proc. ICASSP*, Istanbul, Turkey, June 2000, Vol. V, pp. 3160–3163.

[6] F. Harroy and J.-L. Lacoume, "Maximum Likelihood Estimators and Cramer-Rao Bounds in Source Separation," *Signal Processing*, Vol. 55, No. 2, pp. 167–177, Dec. 1996.

[7] V. Zarzoso and A. K. Nandi, "Blind Separation of Independent Sources for Virtually Any Source Probability Density Function," *IEEE Transactions on Signal Processing*, Vol. 47, No. 9, pp. 2419–2432, Sept. 1999.

[8] V. Zarzoso, A. K. Nandi, F. Herrmann, and J. Millet-Roig, "Combined Estimation Scheme for Blind Source Separation with Arbitrary Source PDFs," *IEE Electronics Letters*, Vol. 37, No. 2, pp. 132–133, Jan. 18, 2001.

[9] M. Ghogho, A. Swami, and T. Durrani, "Approximate Maximum Likelihood Blind Source Separation with Arbitrary Source Pdfs," in *Proc. IEEE SSAP Workshop*, Pocono Manor Inn, PA, Aug. 2000.

[10] F. Herrmann, *Independent Component Analysis with Applications to Blind Source Separation*, Ph.D. thesis, The University of Liverpool, UK, Sept. 2000.

[11] V. Zarzoso and A. K. Nandi, "Unified Formulation of Closed-Form Estimators for Blind Source Separation in Complex Instantaneous Linear Mixtures," in *Proc. EUSIPCO*, Tampere, Finland, Sept. 2000, Vol. I, pp. 597–601.

[12] E. L. Lehmann, *Theory of Point Estimation*, Wadsworth, Inc., Pacific Grove, CA, 1991.